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Stacionární stlačitelné Navier–Stokes–Fourierovy rovnice ve dvou prostorových dimenzích
(Steady compressible Navier–Stokes–Fourier equations in two space dimensions)

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2009

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Abstract: We study steady flow of compressible heat conducting viscous fluid in a bounded two-dimensional domain, described by the Navier–Stokes–Fourier system. We assume that the pressure is given by the constitutive equation $p(\rho,\theta)\ \rho^\gamma + \rho\theta$ for $\gamma > 2$, where ρ is the density and θ is the temperature. We prove existence of a weak solution to these equations without any assumption on the smallness of the data. The proof uses special approximation of the original problem, which guarantees the pointwise boundedness of the density. However, more work has to be done in proof of the strong convergence of the density.
Keywords: steady compressible Navier–Stokes–Fourier equations, slip boundary conditions, weak solutions, large data

1 Introduction

1.1 The Equations

The compressible Navier–Stokes–Fourier system of PDEs (a.k.a. the full Navier–Stokes system) describes steady flow of a compressible heat conducting newtonian fluid in a bounded domain Ω:

$$\begin{aligned} \operatorname{div}(\rho\boldsymbol{v}) &= 0 & (1.1)\\ \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathbf{S}(\boldsymbol{v}) + \nabla p(\rho,\theta) &= \boldsymbol{F} & (1.2)\\ \operatorname{div}(\rho e(\rho,\theta)\boldsymbol{v}) - \operatorname{div}(\kappa(\theta)\nabla\theta) &= \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - p(\rho,\theta)\operatorname{div}\boldsymbol{v}, & (1.3)\end{aligned}$$

where

$$\begin{aligned} \rho : \Omega &\to \mathbb{R}_0^+ \ \ldots \text{density of the fluid (sought)}\\ \boldsymbol{v} : \Omega &\to \mathbb{R}^2 \ \ldots \text{velocity field (sought)}\\ \theta : \Omega &\to \mathbb{R}^+ \ \ldots \text{temperature (sought)}\\ p(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{R}^0 &\to \mathbb{R}_0^+ \ \ldots \text{pressure (given)}\\ \boldsymbol{F} : \Omega &\to \mathbb{R}^2 \ \ldots \text{external force (given)}\\ e(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{R}^+ &\to \mathbb{R}_0^+ \ \ldots \text{internal energy (given)}\end{aligned}$$

$$\begin{aligned} \mathbf{S}(\boldsymbol{v}) &= 2\mu\mathbf{D}(\boldsymbol{v}) + \lambda(\operatorname{div}\boldsymbol{v})\mathbf{I} \ \ldots \text{the viscous part of the stress tensor}\\ \mathbf{D}(\boldsymbol{v}) &= \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T) \ \ldots \text{the symmetric part of the velocity gradient,}\end{aligned}$$

μ, λ are viscosity coefficients (constants) and $\kappa(\theta)$ is heat conductivity.

Note that in full generality equation (1.3) (conservation of internal energy) should be replaced by the conservation of total energy – but for the solution we are about to construct we have balance of kinetic energy as a consequence of the momentum equation[1].

Our solution will be such that $\rho \in L^\infty(\Omega)$ and $\boldsymbol{v} \in W^{1,p}(\Omega)$ for $p < \infty$, we get ($\operatorname{div}(\rho\boldsymbol{v}) = 0$ in a weak sense)

$$\operatorname{div}(\frac{1}{\gamma - 1}\rho^\gamma \boldsymbol{v}) = -\rho^\gamma \operatorname{div} \boldsymbol{v}$$

in a weak sense – so we are allowed to write

$$\operatorname{div}(\rho\theta\boldsymbol{v}) - \operatorname{div}(\kappa(\theta)\nabla\theta) = \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - \rho\theta \operatorname{div} \boldsymbol{v} \tag{1.4}$$

instead of equation (1.3).

1.2 Constitutive equations

We assume that the constitutive equation for the pressure takes the form

$$p(\rho, \theta) = a_1\rho^\gamma + a_2\rho\theta,\ a_1, a_2 > 0, \tag{1.5}$$

i.e. the pressure has one part corresponding to the ideal fluid and a so called elastic part. The internal energy takes the form

$$e(\rho, \theta) = a_3\theta + a_1\frac{\rho^{\gamma-1}}{\gamma - 1},\ a_3 > 0. \tag{1.6}$$

Next we need to assume something about the viscosity coefficients and heat conductivity. So, let μ, λ be constants, let the conditions of thermodynamical stability

$$\mu > 0,\ 3\lambda + 2\mu > 0 \tag{1.7}$$

be satisfied and let κ be such that

$$\kappa(\theta) = a_4(1 + \theta^m),\ a_4, m > 0. \tag{1.8}$$

The constitutive equations are such that the conditions that follow from thermodynamics are fulfilled: the constitutive equations for energy, pressure and temperature fulfill the relation

$$\frac{1}{p^2}(p - \theta\frac{\partial p}{\partial\theta}) = \frac{\partial e}{\partial\rho}, \tag{1.9}$$

[1] To be more specific, we will be able to test the momentum equation by $\boldsymbol{v}$, which gives us also conservation of kinetic energy.

which is straight consequence of the Maxwell relations and which guarantees the existence of entropy; the interested reader should consult the book [NoS].

1.3 Boundary conditions

Let the domain we are working with be sufficiently smooth, i.e. $\Omega \in \mathcal{C}^2$. For the velocity we consider slip boundary conditions:

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot (\mathbf{T}(p, \boldsymbol{v})\boldsymbol{n}) + f\boldsymbol{v} \cdot \boldsymbol{\tau} = 0 \qquad \text{at } \partial\Omega, \tag{1.10}$$

where $\boldsymbol{\tau}$ stands for tangent vector to $\partial\Omega$, $\boldsymbol{n}$ is outer normal vector, $\mathbf{T}(p, \boldsymbol{v}) = -p\mathbf{I} + \mathbf{S}(\boldsymbol{v})$ is the stress tensor and coefficient f is nonnegative constant. (In case $f = 0$ (perfect slip) we need to assume that Ω is not axially symmetric.[2])

For the temperature we assume

$$\kappa(\theta)\frac{\partial \theta}{\partial \boldsymbol{n}} + L(\theta)(\theta - \theta_0) = 0 \text{ at } \partial\Omega, \tag{1.11}$$

where $\theta_0 : \partial\Omega \to \mathbb{R}^+$ is strictly positive sufficiently smooth given function (say $\theta_0 \in \mathcal{C}^2$), $0 < \theta_* \leq \theta_0 \leq \theta^* < \infty$ with θ_*, $\theta^* \in \mathbb{R}^+$ and

$$L(\theta) = a_5(1 + \theta^l), \qquad l \in \mathbb{R}_0^+. \tag{1.12}$$

We must also prescribe total mass of the fluid:

$$\int_\Omega \rho dx = M > 0. \tag{1.13}$$

[2]We need this condition to be fulfilled because of use of Korn's lemma in several moments of the proof.

2 Mathematical background

This section is devoted to some definitions and theorems we would need in the whole work.

2.1 Sobolev spaces

The Sobolev spaces are a very natural tool for the study of PDEs and will be used throughout this whole work.

Definition 2.1 *(Sobolev spaces)*

- *Let* $\Omega \subset \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}_0$, $k \in \mathbb{N}$, $p \in \langle 1, \infty)$. *Sobolev space* $W^{k,p}(\Omega)$ *is defined as (all derivatives are in weak sense):*

  $$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega) \ \forall \alpha : \Sigma \alpha_i \leq k\};$$

  the norm in Sobolev space $W^{k,p}$ *is defined as follows:*

  $$\|u\|_{k,p} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}. \tag{2.1}$$

- *For* $p = \infty$ *we define*

  $$W^{k,\infty}(\Omega) := \{u \in L^\infty(\Omega); D^\alpha u \in L^\infty(\Omega) \ \forall \alpha : \Sigma \alpha_i \leq k\};$$

  the norm in $W^{k,\infty}(\Omega)$ *is defined as*

  $$\|u\|_{k,\infty} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

- *Let* $\Omega \in \mathcal{C}^{k-1,1}$, $q \in \langle 1, \infty)$, $k \in \mathbb{N}$. *Then we are able to define "more exotic" Sobolev space* $W^{k-1/q,q}(\partial\Omega)$ *as a subspace of all functions from* $W^{k-1,q}(\partial\Omega)$ *(*$W^{0,q}(\partial\Omega) = L^q(\partial\Omega)$*) satisfying the following property:*

  $$\forall \alpha, |\alpha| = k-1 : \ I_\alpha(u) = \int_{\partial\Omega} \int_{\partial\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^q}{|x-y|^{n-2+q}} dS_x dS_y < \infty. \tag{2.2}$$

  If we denote

  $$\|u\|_{W^{k-1/q,q}(\partial\Omega)} = \left(\|u\|^q_{W^{[k-1/q],q}(\partial\Omega)} + \sum_{|\alpha|=k-1} I_\alpha(u) \right)^{\frac{1}{q}}, \tag{2.3}$$

  then $\|u\|_{W^{k-1/q,q}(\partial\Omega)}$ *is a norm in* $W^{k-1/q,q}(\partial\Omega)$.

We know that Sobolev space $W^{k,p}(\Omega)$ is a Banach space for all $k \in \mathbb{N}_0$ and $p \in \langle 1, \infty)$. However, the most important properties of Sobolev spaces for us are the imbeddings of such spaces.

Theorem 2.2 *(Imbedding theorems)*
The following assertions hold:

- $1 < p < n$, $\Omega \in \mathcal{C}^{0,1}$: $\forall q \in \langle 1, p^\star \rangle : \; W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.
- $p = n$, $\Omega \in \mathcal{C}^{0,1}$, $\forall q \in \langle 1, \infty) : W^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$.
- $n < p < \infty$, $\Omega \in \mathcal{C}^{0,1}$: $\forall \alpha \in \langle 0, 1 - \frac{n}{p} \rangle : W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$,

where $p^\star = \frac{np}{n-p}$.

Proof. See [KJF], Theorems 5.7.7 and 5.7.8. □

Just for the reader's convenience we remind an inequality which we use in the following: the Poincaré inequality. This inequality can be found in [KJF] as a consequence of Theorem 4.1.1.

Lemma 2.3 *(Poincaré inequality)*
Let $1 \le p < \infty$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain. Let us define*

$$N(u)^p = \left(\int_\Omega |\nabla u|^p dx + c_1 \int_{\Gamma_1} |u|^p d\sigma + c_2 \int_\omega |u|^p dx + c_3 |\int_\Omega u dx|^p \right) \quad (2.4)$$

for $c_i \ge 0$, $c_1 + c_2 + c_3 > 0$, *where* $\omega \subset \Omega$ *is such that* $|\omega|_n > 0$, $\Gamma_1 \subset \partial\Omega$ *is such that* $|\Gamma_1|_{n-1} > 0$.
Then $N(u)$ *is an equivalent norm on* $W^{1,p}(\Omega)$.

We rely on the fact that the reader knows Hölder and Young inequalities, so we do not have to state them here. ☺

During some calculations, we will need the interpolation inequalities:

Lemma 2.4

- *Let* $f \in L^p(\Omega) \cap L^q(\Omega)$, $1 \le p < q \le \infty$. *Then*

$$\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha} \qquad \text{with} \qquad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \; \alpha \in \langle 0, 1 \rangle.$$

- *Let* $f \in W^{1,s}(\Omega) \cap L^q(\Omega)$, $1 \le q < \infty$. *Then*

– *for $s < n$ we have $f \in L^r(\Omega)$, $r \leq \frac{ns}{n-s}$, and for $q \leq r$ it holds:*

$$\|f\|_r \leq \|f\|_{1,s}^{\alpha}\|f\|_q^{1-\alpha} \quad \textit{with} \quad \frac{1}{r} = \alpha(\frac{1}{s} - \frac{1}{n}) + \frac{1-\alpha}{q}, \ \alpha \in \langle 0,1\rangle. \tag{2.5}$$

– *for $s = n$ we are allowed to take $q \leq r < \infty$ and (2.5) holds*

– *for $s > n$ we are allowed to take $r \leq \infty$ and (2.5) holds.*

Proof. The first part is a direct consequence of the Hölder inequality. For the rest, see [M]. □

2.2 Other related topics

In this short subsection we state some results that we will need and appreciate in proofs of future theorems. First of all we will talk about the so called Bogovskii operator, which we will use while proving the a priori estimates to our approximative problem.

Lemma 2.5 *(Bogovskii, Solonnikov, Ladyzhenskaya, Borchers-Sohr, ...) Let $\Omega \in \mathcal{C}^{0,1}$ be a bounded domain, $\Omega \subset \mathbb{R}^n$; let $f \in W_0^{m,q}(\Omega)$ for $m \geq 0$, $1 < q < \infty$, $\int_\Omega f = 0$. Then there exists $\boldsymbol{u} \in (W_0^{m+1,q}(\Omega))^n$ which solves*

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= f \qquad \textit{in } \Omega \\ \boldsymbol{u} &= 0 \qquad \textit{at } \partial\Omega \end{aligned}$$

such that

$$\|\nabla \boldsymbol{u}\|_{m,q} \leq C\|f\|_{m,q}.$$

Proof. See [NoS], Auxiliary lemma 3.15. □

This little definition (and the following theorem) we will strongly appreciate when proving existence to the approximative problem.

Definition 2.6 *Let $(X, \|\cdot\|_X)$ be a Banach space and $\Omega \subset X$ bounded open set. We say that $F : \overline{\Omega} \times \langle 0,1\rangle \to X$ is homotopy of compact transformations on $\overline{\Omega}$ if*

1. *$F(\cdot,t) : \overline{\Omega} \to X$ is compact operator for any $t \in \langle 0,1\rangle$*

2. for any $\kappa_1 > 0$ and for any $\widetilde{\Omega} \subset \Omega$ there exists $\kappa_2 > 0$:

$$\|F(x,t) - F(x,s)\| < \kappa_1 \quad \forall x \in \widetilde{\Omega}\ \forall s,t \in \langle 0,1\rangle,\ |s-t| < \kappa_2$$

Theorem 2.7 *(Leray–Schauder fixed-point theorem)*
Let X be a Banach space, $\Omega \subset X$ a bounded open set; let $F : \overline{\Omega} \times \langle 0,1\rangle \to X$ be a homotopy of compact transformations on $\overline{\Omega}$ such that

$$0 \notin (\mathbf{I} - F(\cdot,t))(\partial\Omega), \quad t \in \langle 0,1\rangle.$$

If there exists at least one $u_0 \in \Omega$ such that $F(u_0, 0) = u_0$, then there exists at least one solution $u_t \in \Omega$ to the problem

$$H(u_t, t) = u_t$$

for any $t \in \langle 0,1\rangle$.

Proof. This theorem follows directly from Theorems 7.8 and 7.10 in [FoG]. □

Next, we move virtually into the Section 5 to examine bounded sequences in Banach spaces.

Theorem 2.8 *(Eberlein–Schmulyan)*
Let X be a reflexive Banach space and let $\{u_n\} \subset X$ be a bounded sequence. Then there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ weakly convergent in X.

Theorem 2.9 *(Banach–Alaoglu)*
Let X be a separable Banach space and let $\{f_n\} \subset X^*$ be a bounded sequence. Then there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ weakly-* convergent in X^*.

The next theorem will help us to prove some facts about convergence of the temperature.

Theorem 2.10 *(Vitali)*
Let $f_n \to f$ in some measurable set $M \subset \mathbb{R}^n$. Then $f_n \to f$ strongly in $L^1(M)$ iff the following two conditions hold true:

1. For any $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$\int_E |f_n| < \varepsilon$$

for $n \in \mathbb{N}$, $E \subset M$ such that $|E| < \delta$.

2. For any ε there exists $E_\varepsilon \subset M$ of finite measure such that

$$\int_{M \setminus E_\varepsilon} |f_n| < \varepsilon$$

for $n \in \mathbb{N}$.

The last lemma in this section we will find very useful in the proof of convergence of the densiy.

Lemma 2.11 *Let $\Omega \in \mathcal{C}^{0,1}$, $\boldsymbol{v} \in W^{1,q}(\Omega)$, $1 < q < \infty$, $\rho \in L^p(\Omega)$, $1 < p < \infty$, $\boldsymbol{v} \cdot \nabla \rho \in L^s(\Omega)$, $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. Then there exists $\rho_n \in \mathcal{C}^\infty(\overline{\Omega})$ such that*

$$\boldsymbol{v} \cdot \nabla \rho_n \to \boldsymbol{v} \cdot \nabla \rho \qquad \text{in } L^s(\Omega)$$

and

$$\rho_n \to \rho \qquad \text{in } L^p(\Omega).$$

Proof. See [NNP], Theorem 1.1. □

3 Theorem

3.1 Weak formulation & Weak solution

Definition 3.1 *The triple $(\rho, \boldsymbol{v}, \theta)$ is called a weak solution to problem (1.1)–(1.13) if $\rho \in L^\infty(\Omega)$, $\boldsymbol{v} \in W^{1,2}(\Omega)$, $\theta \in W^{1,2}(\Omega)$ and $\theta^m \nabla\theta \in L^1(\Omega)$, $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ at $\partial\Omega$ in a trace sense and*

$$\int_\Omega \rho\boldsymbol{v}\cdot\nabla\eta = 0 \qquad \forall \eta \in \mathcal{C}^\infty(\overline{\Omega}) \tag{3.1}$$

$$\begin{gathered}\int_\Omega (-\rho\boldsymbol{v}\otimes\boldsymbol{v} : \nabla\boldsymbol{\varphi} + 2\mu\mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi}) + \lambda \operatorname{div}\boldsymbol{v} \operatorname{div}\boldsymbol{\varphi} - p(\rho,\theta)\operatorname{div}\boldsymbol{\varphi})\, dx + \\ f\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{\varphi} d\sigma = \int_\Omega \rho\boldsymbol{F}\cdot\boldsymbol{\varphi} dx \qquad \forall\boldsymbol{\varphi} \in \mathcal{C}^2(\overline{\Omega}); \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \text{ at } \partial\Omega\end{gathered} \tag{3.2}$$

$$\begin{gathered}\int_\Omega (\kappa(\theta)\nabla\theta\cdot\nabla\psi - \rho\theta\boldsymbol{v}\nabla\psi)dx + \int_{\partial\Omega} L(\theta)(\theta-\theta_0)\psi d\sigma = \\ \int_\Omega (2\mu|\mathbf{D}(\boldsymbol{v})|^2\psi + \lambda(\operatorname{div}\boldsymbol{v})^2\psi - \rho\theta\operatorname{div}\boldsymbol{v}\psi)dx \qquad \forall\psi\in\mathcal{C}^\infty(\overline{\Omega}).\end{gathered} \tag{3.3}$$

3.2 The Theorem

Now we are ready to state our main result:

Theorem 3.2 *Let $\Omega \in \mathcal{C}^2$ be a bounded domain, $\Omega \subset \mathbb{R}^2$. Let $\boldsymbol{F} \in L^\infty(\Omega)$, $m = l + 1$ and*

$$\gamma > 2, \qquad m > \frac{\gamma - 1}{\gamma - 2}.$$

Then there exists a weak solution to (1.1)–(1.13) such that $\rho \in L^\infty(\Omega)$, $\boldsymbol{v} \in W^{1,q}(\Omega)$ and $\theta \in W^{1,q}(\Omega)$ for all $1 \le q < \infty$.

Note that the Theorem could be proved also for $m \neq l+1$ and $\boldsymbol{F} \in L^p(\Omega)$ for $p < \infty$; however, the details of the proof would be much more complicated than in our "simple" case. The interested reader may find the conditions for γ, m, $l+1$ and p in (4.40).

3.3 Proof: history & some basic hints

The aim of this work is to prove Theorem 3.2. We will continue in work of Mucha and Pokorný ([MP] and [PM] – 2D and 3D Navier–Stokes equations, no temperature or internal energy is considered here; [MP2] and [MP3] – 3D Navier–Stokes–Fourier equations, temperature and equation for internal energy included). The achievements of these works were proofs for $\gamma > 3$ and $m > \frac{3\gamma-1}{3\gamma-7}$ in [MP2] and $\gamma > \frac{7}{3}$ and $m > \frac{3\gamma-1}{3\gamma-7}$ in [MP3] (here $m = l+1$); our results are for $\gamma > 2$ and $m > \frac{\gamma-1}{\gamma-2}$, which is, in some way, better result than in 3D, but not better than in 2D without temperature ([MP]).

One of the possible approaches to the problem (1.1)–(1.13) was introduced in [L]; unfortunately, the author considered $\int_\Omega \rho^p = M^p$ for sufficiently large p instead of (1.13), which is not really acceptable from the physical point of view.

To prove Theorem 3.2 we have to construct an approximation of the original problem, which we implement in Section 4. We prove some a priori estimates for the approximative system (Subsection 4.1) and then we prove existence of a solution to the approximative system (Subsection 4.2). To that purpose we define two interesting compact operators and then we use Leray–Schauder fixed point theorem (Theorem 2.7) to show that the operators work in a way we want them. The last thing we prove in Subsection 4.2 are some estimates for density, velocity and temperature.

In the next section (Section 5) we have to show that the solution to the approximative problem we constructed in the previous section converges to the solution of the original system, at least in some sense. To this purpose, we prove that the temperature is no problem for convergence and then we introduce quantity known as effective viscous flux, which plays key role in proof of convergence of density. The last difficulty remains velocity, but once the density is solved, this would be no problem for us.

4 Proof I: Approximation

We are going to construct the approximation of the original problem in the same way as in [MP2]. We need constant $\varepsilon > 0$ and a function $K(t)$ with

$$K(t) = \begin{cases} 1 & \text{for } t < k \\ \in \langle 0, 1 \rangle & \text{for } k \le t \le k+1 \\ 0 & \text{for } t > k+1, \end{cases} \tag{4.1}$$

$k > 0$. Moreover we assume that $K'(t) < 0$ for $t \in (k, k+1)$. Later we pass $\varepsilon \to 0+$ and in that moment we would need to know that $K(\rho) \equiv 1$ for sufficiently large value of k.

In Ω we have:

$$\varepsilon\rho + \operatorname{div}(K(\rho)\rho\boldsymbol{v}) - \varepsilon\Delta\rho = \varepsilon h K(\rho) \tag{4.2}$$

$$\frac{1}{2}\operatorname{div}(K(\rho)\rho\boldsymbol{v}\otimes\boldsymbol{v}) + \frac{1}{2}K(\rho)\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \operatorname{div}\mathbf{S}(\boldsymbol{v}) + \nabla P(\rho,\theta) = K(\rho)\rho\boldsymbol{F} \tag{4.3}$$

$$-\operatorname{div}\left((1+\theta^m)\frac{\varepsilon+\theta}{\theta}\nabla\theta\right) + \operatorname{div}\left(\boldsymbol{v}\int_0^\rho K(t)dt\right)\theta + \operatorname{div}\left(K(\rho)\rho\boldsymbol{v}\right)\theta + K(\rho)\rho\boldsymbol{v}\cdot\nabla\theta - \theta K(\rho)\boldsymbol{v}\cdot\nabla\rho = \mathbf{S}(\boldsymbol{v}) : \nabla(\boldsymbol{v}), \tag{4.4}$$

where

$$P(\rho,\theta) = \int_0^\rho \gamma t^{\gamma-1}K(t)dt + \theta\int_0^\rho K(t)dt = P_b(\rho) + \theta\int_0^\rho K(t)dt \tag{4.5}$$

and $h = \frac{M}{\Omega}$.

If we define "entropy" s as

$$s = \ln\theta.$$

If we consider θ sufficiently smooth, we may rewrite equation (4.4) as

$$-\operatorname{div}\left((1+e^{sm})\frac{\varepsilon+e^s}{e^s}\nabla s\right) + K(\rho)\rho\boldsymbol{v}\cdot\nabla s - K(\rho)\boldsymbol{v}\cdot\nabla\rho + \operatorname{div}\left(\boldsymbol{v}\int_0^\rho K(t)dt\right) + \operatorname{div}\left(K(\rho)\rho\boldsymbol{v}\right) = \frac{\mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v}}{e^s} + \frac{(1+e^{sm})(\varepsilon+e^s)}{e^s}|\nabla s|^2 \tag{4.6}$$

in Ω. We will appreciate this form of internal energy equation (or rather entropy equation) later – we will need to control the positivness of temperature, which does not work very good with equation (4.4).

(Note that s is not entropy in thermodynamical sense, although it behaves like entropy – for us, this is just auxiliary function. The "real" entropy would consist of a part depending on θ as well as of some contribution from the elastic part of the pressure function.)

The boundary conditions (see Subsection 1.3 for original BCs) we consider as follows:

$$(1+\theta^m)(\varepsilon+\theta)\frac{\partial s}{\partial \boldsymbol{n}} + L(\theta)(\theta-\theta_0) + \varepsilon s = 0 \tag{4.7}$$

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{\tau}\cdot(\mathbf{T}(p,\boldsymbol{v})\boldsymbol{n}) + f\boldsymbol{v}\cdot\boldsymbol{\tau} = 0 \tag{4.8}$$

$$\frac{\partial \rho}{\partial \boldsymbol{n}} = 0. \tag{4.9}$$

4.1 A priori estimates for the approximative system

Lemma 4.1 *Let us suppose that the solution is sufficiently smooth, i.e. ρ, $\boldsymbol{v}$, $\theta \in W^{2,q}(\Omega)$ for any $q<\infty$, $\theta>0$. Let assumptions of Theorem 3.2 be satisfied. Then*

$$0 \le \rho \le k+1, \qquad \int_\Omega \rho dx \le M,$$

$$\begin{aligned} \|\boldsymbol{v}\|_{1,2} + \|K(\rho)\rho\|_{2\gamma} + \|P(\rho,\theta)\|_2 + \|\theta\|_q + \|\nabla\theta\|_{1+\delta} + \\ \int_{\partial\Omega}(e^s+e^{-s})d\sigma + \|\nabla s\|_2 \le C(\|\boldsymbol{F}\|_\infty, M, q), \end{aligned} \tag{4.10}$$

where $\delta = 1$ for $m\ge 2$, $\delta<1$ for $m<2$ and the RHS of (4.10) is independent of ε and k, $s=\ln\theta$.

Proof.
First of all we will state main ideas of the proof of non-negativeness and the boundedness of the density. First, we integrate the approximative continuity equation (4.2) over Ω. After using Gauss–Ostrogradskii theorem and the boundary conditions we get

$$\int_\Omega \varepsilon\rho dx = \int_\Omega \varepsilon h K(\rho) dx,$$

from where we have L^1 bound on the density. Next we denote by Ω_- subdomain where $\rho \leq 0$ and Ω_k subdomain where $\rho > k+1$. We integrate over Ω_-:

$$\int_{\Omega_-} \varepsilon\rho dx + \int_{\partial\Omega_-} K(\rho)\rho\boldsymbol{v}\cdot\boldsymbol{n}d\sigma - \varepsilon\int_{\partial\Omega_-}\frac{\partial\rho}{\partial\boldsymbol{n}}d\sigma = \varepsilon h\int_{\Omega_-}K(\rho).$$

Now we realize that due to the regularity of ρ and $\boldsymbol{v}$ and due to the boundary conditions the surface integrals disappear; this, together with properties of $K(\cdot)$ implies $|\{x\in\Omega;\rho(x)<0\}| = 0$. Similarly we prove $\rho(x) < k+1$.
Details can be found in [PM] (3D case is considered here) or [MP]; for another approach see [NoS].

In what follows we prove (4.10). Let us assume a little generalization for a while: let's consider $\boldsymbol{F}\in L^p(\Omega)$ only. We will show some a priori estimates for such generalized RHS and then we will put $\boldsymbol{F}\in L^\infty(\Omega)$ and $m = l+1$ to simplify further calculations. (The existence of the solution can be shown also in these "complicated" cases – we give some comments on this topic later – but it would be with many technical problems; see (4.40).)

First of all, we multiply approximative momentum equation (4.3) by $\boldsymbol{v}$ and integrate over Ω. We get

$$\begin{aligned}\int_\Omega \left(2\mu\mathbf{D}^2(\boldsymbol{v}) + \lambda\mathrm{div}^2\boldsymbol{v}\right)dx + \int_{\partial\Omega} f|\boldsymbol{v}\odot\boldsymbol{\tau}|^2 d\sigma + \int_\Omega \boldsymbol{v}\cdot\nabla P_b(\rho)dx = \\ \int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\boldsymbol{F}dx + \int_\Omega\Big(\int_0^\rho K(t)dt\Big)\theta\,\mathrm{div}\,\boldsymbol{v}dx.\end{aligned} \tag{4.11}$$

To get information about the last term on the LHS of this equation we use approximative continuity equation (4.2):

$$\begin{aligned}\int_\Omega \boldsymbol{v}\cdot\nabla P_b(\rho)dx = \frac{\gamma}{\gamma-1}\int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\nabla\rho^{\gamma-1}dx = \\ -\frac{\gamma}{\gamma-1}\int_\Omega\left(\varepsilon\Delta\rho + \varepsilon hK(\rho) - \varepsilon\rho\right)\rho^{\gamma-1}dx = \\ \frac{\varepsilon\gamma}{\gamma-1}\int_\Omega(\rho - hK(\rho))\rho^{\gamma-1}dx + \varepsilon\gamma\int_\Omega\rho^{\gamma-2}|\nabla\rho|^2dx.\end{aligned}$$

Using this term and the momentum equation we get

$$\begin{aligned}\int_\Omega \mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}dx + \int_{\partial\Omega} f|\boldsymbol{v}\odot\boldsymbol{\tau}|^2d\sigma + \varepsilon\gamma\int_\Omega\rho^{\gamma-2}|\nabla\rho|^2dx + \frac{\varepsilon\gamma}{\gamma-1}\int_\Omega\rho^\gamma dx - \\ \int_\Omega\Big(\int_0^\rho K(t)dt\Big)\theta\,\mathrm{div}\,\boldsymbol{v}dx \leq C\Big(1 + \int_\Omega|K(\rho)\rho\boldsymbol{v}\cdot\boldsymbol{F}|dx\Big).\end{aligned} \tag{4.12}$$

Integrating the energy equation (4.4) and including information from the boundary condition we get

$$\int_{\partial\Omega} \big(L(\theta)(\theta-\theta_0)+\varepsilon s\big)d\sigma = \int_\Omega \Big(\mathbf{S}(\boldsymbol{v}) : \nabla \boldsymbol{v} - \big(\int_0^\rho K(t)dt\big)\theta \operatorname{div} \boldsymbol{v}\Big)dx, \quad (4.13)$$

since the integration by parts gives the following identity

$$\begin{aligned}\int_\Omega \Big(K(\rho)\rho\boldsymbol{v}\cdot\nabla\theta - \theta K(\rho)\boldsymbol{v}\cdot\nabla\rho + \operatorname{div}\Big(\boldsymbol{v}\int_0^\rho K(t)dt\Big)\theta + \\ \operatorname{div}\big(K(\rho)\rho\boldsymbol{v}\big)\theta\Big)dx = \int_\Omega \Big(\int_0^\rho K(t)dt\Big)\theta \operatorname{div}\boldsymbol{v}dx.\end{aligned} \quad (4.14)$$

Summing up (4.12) and (4.13) we get

$$\begin{aligned}\int_{\partial\Omega} \big(L(\theta)\theta + \varepsilon s^+\big)d\sigma + \varepsilon\gamma\int_\Omega \rho^{\gamma-2}|\nabla\rho|^2dx + \frac{\varepsilon\gamma}{\gamma-1}\int_\Omega \rho^\gamma dx \leq \\ \int_{\partial\Omega}\varepsilon s^- d\sigma + C\Big(1+\int_\Omega |K(\rho)\rho\boldsymbol{v}\cdot\boldsymbol{F}|dx\Big),\end{aligned} \quad (4.15)$$

where s^+ and s^- are the positive and negative parts of the entropy ($s = s^+ - s^-$). Now we take care of the first term of the RHS of (4.15). Note that the control of the negative part of entropy s is not immediate.

We integrate the entropy equation (4.6) over Ω getting

$$\begin{aligned}\int_{\partial\Omega}\Big(\frac{L(\theta)(\theta-\theta_0)}{\theta} + \varepsilon s e^{-s}\Big)d\sigma + \int_\Omega \Big(K(\rho)\rho\frac{\boldsymbol{v}\cdot\nabla\theta}{\theta} - K(\rho)\boldsymbol{v}\cdot\nabla\rho\Big)dx = \\ \int_\Omega\Big(\frac{\mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}}{\theta} + \frac{(1+\theta^m)(\varepsilon+\theta)}{\theta}|\nabla s|^2\Big)dx.\end{aligned} \quad (4.16)$$

So

$$\begin{aligned}\int_\Omega\Big(\frac{\mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}}{\theta} + \frac{(1+\theta^m)(\varepsilon+\theta)}{\theta}|\nabla s|^2\Big)dx + \int_{\partial\Omega}\Big(\frac{L(\theta)\theta_0}{\theta} + \varepsilon|s^-|e^{|s^-|}\Big)d\sigma \\ -\int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\nabla(s-\ln\rho)dx \leq \int_{\partial\Omega}L(\theta)d\sigma + \int_{\partial\Omega}\varepsilon s^+e^{-s^+}d\sigma.\end{aligned} \quad (4.17)$$

From the last term in the LHS of (4.17) we have

$$-\int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\nabla(s-\ln\rho)dx = \int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\nabla\ln\rho dx - \int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\nabla s dx \quad (4.18)$$

and with help of (4.2) we get for the first integral in (4.18)

$$\begin{aligned}\int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\nabla\ln\rho dx = -\int_\Omega \operatorname{div}(K(\rho)\rho\boldsymbol{v})\ln\rho dx = \\ \int_\Omega \big(-\varepsilon\Delta\rho+\varepsilon\rho-\varepsilon hK(\rho)\big)\ln\rho dx = \\ \int_\Omega \Big(\varepsilon\frac{|\nabla\rho|^2}{\rho}-\varepsilon hK(\rho)\ln\rho+\varepsilon\rho\ln\rho\Big)dx.\end{aligned} \tag{4.19}$$

The first term has a good sign, the second term has a good sign for $\rho\le 1$, too, and for $\rho\ge 1$ is easily bounded by $\varepsilon h\rho$. Similarly, the last term can be controlled by the term $\varepsilon\int_\Omega\rho^\gamma dx$. The proof was rather formal, as we do not know whether $\rho>0$ in Ω. However, we may write $K(\rho)\boldsymbol{v}\cdot\nabla(\rho+\delta)$ in (4.18) with $\delta>0$ and find an analogue of (4.19) with $\ln(\rho+\delta)$. Finally we pass with $\delta\to 0^+$ and get precisely the same information as above.

Next, for the second integral in (4.18):

$$\begin{aligned}-\int_\Omega K(\rho)\rho\boldsymbol{v}\cdot\nabla s dx = \int_\Omega \big(\varepsilon\Delta\rho-\varepsilon\rho+\varepsilon hK(\rho)\big)s dx = \\ \int_\Omega \big(-\varepsilon\nabla\rho\nabla s-\varepsilon\rho\ln\theta+\varepsilon hK(\rho)\ln\theta\big)dx.\end{aligned} \tag{4.20}$$

Considering the RHS of (4.20), we have

$$\begin{aligned}\left|\varepsilon\int_\Omega\nabla\rho\nabla s dx\right| \le \varepsilon\|\nabla\rho\|_2\|\nabla s\|_2 \le \\ \frac{1}{4}\varepsilon\Big(\int_\Omega\frac{|\nabla\rho|^2}{\rho}dx+\int_\Omega|\nabla\rho|^2\rho^{\gamma-2}dx\Big)+\frac{1}{4}\|\nabla s\|^2_{L_2(\Omega)}.\end{aligned} \tag{4.21}$$

Moreover, $\int_\Omega -\varepsilon\rho\ln\theta dx$ has a good sign for $\theta\le 1$; for $\theta>1$ we have

$$\begin{aligned}\int_\Omega -\varepsilon\rho(\ln\theta)^+dx \le \varepsilon\|\rho\|_2\|s^+\|_2 \le \\ \frac{\varepsilon}{4}\big(\|s^+\|_{L_1(\partial\Omega)}+\|\nabla s\|_2\big)+\frac{\varepsilon}{4}\|\rho^\gamma\|_1+C.\end{aligned} \tag{4.22}$$

The last term of (4.20) can be treated as follows (one part has again a good sign)

$$\int_\Omega \varepsilon hK(\rho)|(\ln\theta)^-|dx \le C\varepsilon\int_\Omega|s^-|dx \le C+\frac{1}{2}\int_{\partial\Omega}\varepsilon|s^-|e^{|s^-|}d\sigma+\frac{1}{4}||\nabla s||_{L_2(\Omega)}. \tag{4.23}$$

Then combining (4.17) with inequality (4.15) and with (4.19)–(4.23) we obtain

$$\int_\Omega \left(\frac{\mathbf{S}(\boldsymbol{v}) : \nabla \boldsymbol{v}}{\theta} + \frac{1+\theta^m}{\theta^2} |\nabla \theta|^2 \right) dx + \int_{\partial\Omega} \left(L(\theta)\theta + \frac{L(\theta)\theta_0}{\theta} + \varepsilon |s| \right) d\sigma \le H, \tag{4.24}$$

where

$$H = C\Big(1 + \int_\Omega |K(\rho)\rho \boldsymbol{v} \cdot \boldsymbol{F}| dx\Big).$$

From the growth conditions we deduce:

$$\left(\int_{\partial\Omega} \theta^{l+1} d\sigma \right)^{1/(l+1)} \le H^{1/(l+1)}, \qquad \left(\int_\Omega |\nabla \theta^{m/2}|^2 \right)^{1/m} \le H^{1/m}.$$

To get some bounds on temperature we use the following Poincaré type inequality:

$$\left(\int_\Omega |\theta^{m/2}|^2 dx \right)^{1/m} \le C(\Omega) \left(\left(\int_\Omega |\nabla \theta^{m/2}|^2 dx \right)^{1/m} + \left(\int_{\partial\Omega} \theta^{l+1} d\sigma \right)^{1/(l+1)} \right).$$

The imbedding theorem leads to the bound

$$\left(\int_\Omega \theta^q dx \right)^{1/q} \le C \left(H^{1/m} + H^{1/(l+1)} \right). \tag{4.25}$$

for any $q < \infty$; note that $C = C(q)$ and $C(q) \to \infty$ for $q \to \infty$.

We now return to (4.12). Hölder's inequality with help of Korn's inequality[3] yields

$$\begin{aligned} &\|\boldsymbol{v}\|_{1,2}^2 + \varepsilon\gamma \int_\Omega \rho^{\gamma-2} dx + \frac{\varepsilon\gamma}{\gamma-1} \int_\Omega \rho^\gamma dx \le \\ C &\left(1 + \int_\Omega |K(\rho)\rho \boldsymbol{v} \cdot \boldsymbol{F}| dx + \int_\Omega |\theta \int_0^\rho K(t) dt|^2 dx \right). \end{aligned} \tag{4.26}$$

In what follows we use the imbedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in \langle 1, \infty)$; in the following q is always this number taken from the imbedding. (Note that our aim is to use q as large as possible.)

[3] i.e. for $f = 0$ we require that Ω is not rotationally symmetric, for more details see [NoS]

The first term on the RHS of this equation is simple (we only need $\frac{pq}{pq-p-q} \in (1, 2\gamma)$):

$$\int_\Omega |K(\rho)\rho\boldsymbol{v} \cdot \boldsymbol{F}|dx \leq C\|K(\rho)\rho\|_{\frac{pq}{pq-p-q}}\|\boldsymbol{v}\|_q\|\boldsymbol{F}\|_p \leq C\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(p+q)}{pq(2\gamma-1)}}\|\boldsymbol{v}\|_{1,2}. \tag{4.27}$$

However, for successful estimation of the rest of RHS of (4.26) we will have to get bound of $P_b(\rho)$. To find it we use the so-called Bogovskii operator. We introduce $\boldsymbol{\Phi} : \Omega \to \mathbb{R}^2$ defined as a solution to the following problem (here $\{P_b(\rho)\} = \frac{1}{|\Omega|}\int_\Omega P_b(\rho)dx$):

$$\begin{aligned} \operatorname{div} \boldsymbol{\Phi} &= P_b(\rho) - \{P_b(\rho)\} \text{ in } \Omega \\ \boldsymbol{\Phi} &= \mathbf{0} \text{ at } \partial\Omega. \end{aligned} \tag{4.28}$$

From properties of this operator (see Subsection 2.2) we get

$$\|\boldsymbol{\Phi}\|_{W_0^{1,2}(\Omega)} \leq C\|P_b(\rho)\|_2 \tag{4.29}$$

and then, using structure of $P_b(\rho)$, the fact that $\int_\Omega \rho dx \leq M$ and interpolation inequality:

$$\{P_b(\rho)\} \leq \delta\|P_b(\rho)\|_2 + C(\delta, M) \qquad \text{for all } \delta > 0.$$

Now, we are ready to multiply the approximative momentum equation (4.3) by $\boldsymbol{\Phi}$, use (4.26) and (4.29), use some standard estimates on RHS to (4.3) and get

$$\|P_b(\rho)\|_2^2 \leq C\left(1 + \int_\Omega |K(\rho)\rho\boldsymbol{v} \otimes \boldsymbol{v}|^2 dx + \int_\Omega |\theta\int_0^\rho K(t)dt|^2 dx\right). \tag{4.30}$$

As

$$\|P_b(\rho)\|_2^2 \geq C\left(\int_\Omega (K(\rho)\rho)^{2\gamma}dx + \int_\Omega \left(\int_0^\rho K(t)dt\right)^{2\gamma}\right), \tag{4.31}$$

we get this nice bound for the first integral in RHS of (4.30):

$$\int_\Omega |K(\rho)\rho\boldsymbol{v} \otimes \boldsymbol{v}|^2 dx \leq C\|K(\rho)\rho\|_{\frac{2q}{q-4}}^2\|\boldsymbol{v}\|_q^4 \leq \|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(q+4)}{q(2\gamma-1)}}\|\boldsymbol{v}\|_q^4 \tag{4.32}$$

for any $q < \infty$. We now take care of the second integral in (4.30) with a help of Hölder's inequality; we get $\|\int_0^\rho K(t)dt\|_{\frac{2q}{q-2}}$ and then we use interpolation between L^1 and $L^{2\gamma}$ (for the interpolation we require $\frac{2q}{q-2} \in (1, 2\gamma)$ – but for γ which satisfies assumptions from Theorem 3.2 we have this for free):

$$\|\theta \int_0^\rho K(t)dt\|_2^2 \leq \|\theta\|_q^2 \| \int_0^\rho K(t)dt\|_{\frac{2q}{q-2}}^2 \leq \|\theta\|_q^2 \| \int_0^\rho K(t)dt\|_{2\gamma}^{\frac{2\gamma(q+2)}{q(2\gamma-1)}}. \quad (4.33)$$

Apart from that, we have (4.25) and (4.27), which gives us the following bound:

$$\|\theta\|_q \leq \left(\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(p+q)}{pq(2\gamma-1)}} \|\boldsymbol{v}\|_{1,2}\right)^{\frac{1}{m}} + \left(\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(p+q)}{pq(2\gamma-1)}} \|\boldsymbol{v}\|_{1,2}\right)^{\frac{1}{l+1}}. \quad (4.34)$$

Therefore, we get

$$\|\theta \int_0^\rho K(t)dt\|_2^2 \leq \| \int_0^\rho K(t)dt\|_{2\gamma}^{\frac{2\gamma(q+2)}{q(2\gamma-1)}} \Big(\left(\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(p+q)}{pq(2\gamma-1)}} \|\boldsymbol{v}\|_{1,2}\right)^{\frac{2}{m}} + \left(\|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(p+q)}{pq(2\gamma-1)}} \|\boldsymbol{v}\|_{1,2}\right)^{\frac{2}{l+1}} \Big). \quad (4.35)$$

From the inequalities above we see that

$$\|K(\rho)\rho\|_{2\gamma}^{2\gamma} \leq \|K(\rho)\rho\|_{2\gamma}^{\frac{2\gamma(q+4)}{q(2\gamma-1)}} \|\boldsymbol{v}\|_{1,2}^4 + \|K(\rho)\rho\|_{2\gamma}^{2\gamma\left(\frac{q+2}{q(2\gamma-1)} + \frac{2}{m}\frac{p+q}{pq(2\gamma-1)}\right)} \|\boldsymbol{v}\|_{1,2}^{\frac{2}{m}} + \|K(\rho)\rho\|_{2\gamma}^{2\gamma\left(\frac{q+2}{q(2\gamma-1)} + \frac{2}{l+1}\frac{p+q}{pq(2\gamma-1)}\right)} \|\boldsymbol{v}\|_{1,2}^{\frac{2}{l+1}}. \quad (4.36)$$

Young's inequality yields

$$\|K(\rho)\rho\|_{2\gamma}^{2\gamma} \leq \|\boldsymbol{v}\|_{1,2}^{\frac{2q(2\gamma-1)}{q(\gamma-1)-2}} + \|\boldsymbol{v}\|_{1,2}^{\frac{pq(2\gamma-1)}{mpq(\gamma-1)-mp-(p+q)}} + \|\boldsymbol{v}\|_{1,2}^{\frac{pq(2\gamma-1)}{(l+1)pq(\gamma-1)-(l+1)p-(p+q)}}, \quad (4.37)$$

which, together with (4.24) and a little help of Young's inequality, leads us to the final bound

$$\begin{aligned}\|\boldsymbol{v}\|_{1,2}^2 \leq{}& \|\boldsymbol{v}\|_{1,2}^{1+\frac{p+q}{pq(2\gamma-1)}\frac{2q(2\gamma-1)}{q(\gamma-1)-2}} + \|\boldsymbol{v}\|_{1,2}^{1+\frac{p+q}{pq(2\gamma-1)}\frac{pq(2\gamma-1)}{mpq(\gamma-1)-mp-(p+q)}} + \\ & \|\boldsymbol{v}\|_{1,2}^{1+\frac{p+q}{pq(2\gamma-1)}\frac{pq(2\gamma-1)}{(l+1)pq(\gamma-1)-(l+1)p-(p+q)}} + \|\boldsymbol{v}\|_{1,2}^{\frac{2q(2\gamma-1)}{q(\gamma-1)-2}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{m}\frac{p+q}{pq(2\gamma-1)}\right)+\frac{2}{m}} + \\ & \|\boldsymbol{v}\|_{1,2}^{\frac{2q(2\gamma-1)}{q(\gamma-1)-2}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{l+1}\frac{p+q}{pq(2\gamma-1)}\right)+\frac{2}{l+1}} + \|\boldsymbol{v}\|_{1,2}^{\frac{pq(2\gamma-1)}{mpq(\gamma-1)-mp-(p+q)}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{m}\frac{p+q}{pq(2\gamma-1)}\right)+\frac{2}{m}} + \\ & \|\boldsymbol{v}\|_{1,2}^{\frac{pq(2\gamma-1)}{(l+1)pq(\gamma-1)-(l+1)p-(p+q)}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{l+1}\frac{p+q}{pq(2\gamma-1)}\right)+\frac{2}{l+1}} + \\ & \|\boldsymbol{v}\|_{1,2}^{\frac{pq(2\gamma-1)}{(l+1)pq(\gamma-1)-(l+1)p-(p+q)}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{m}\frac{p+q}{pq(2\gamma-1)}\right)+\frac{2}{m}} + \\ & \|\boldsymbol{v}\|_{1,2}^{\frac{pq(2\gamma-1)}{mpq(\gamma-1)-mp-(p+q)}\left(\frac{q+2}{q(2\gamma-1)}+\frac{2}{l+1}\frac{p+q}{pq(2\gamma-1)}\right)+\frac{2}{l+1}}.\end{aligned} \quad (4.38)$$

To get a reasonable bound on $\|\boldsymbol{v}\|_{1,2}$ we need all the exponents in (4.38) to be less than 2. After some algebra we get

$$
\begin{aligned}
&\gamma > 2\\
&\gamma > \frac{4p+2q+pq}{pq}\\
&m > \frac{2(p+q)}{p(q(\gamma-1)-1)} \text{ and the same term for } l+1\\
&m > \frac{pq-4q+2pq(\gamma-1)}{2q(p(\gamma-1)-1)} \text{ and the same term for } l+1\\
&m > \frac{(l+1)p(q+2)+2(p+q)+2l(p+q)}{2lp(q(\gamma-1)-1)}\\
&m > \frac{2q+pq(\gamma-1)}{pq(\gamma-1)-4p-pq} \text{ and the same term for } l+1;
\end{aligned}
\tag{4.39}
$$

the same result, but for $q \to \infty$ (which seems rather formal, but as we are interested in q as big as possible, this limit gives us the right conditions on γ and m) is

$$
\begin{aligned}
&\gamma > 2\\
&\gamma > \frac{p+2}{p}\\
&m > \frac{2}{p(\gamma-1)} \text{ and the same term for } l+1\\
&m > \frac{p(2\gamma-1)-4}{p(\gamma-1)-4} \text{ and the same term for } l+1\\
&m > \frac{p(l+1)+2+2l}{2lp(\gamma-1)}\\
&m > \frac{2+p(\gamma-1)}{p(\gamma-2)} \text{ and the same term for } l+1.
\end{aligned}
\tag{4.40}
$$

For the sake of simplicity we will consider $p = \infty$ and $m = l+1$ only – we want to avoid unnecessary technical problems; applying these new restrictions, we get simple conditions for m and γ:

$$\gamma > 2 \qquad \& \qquad m > \frac{\gamma-1}{\gamma-2}. \tag{4.41}$$

Now we are almost done: we have

$$\|\boldsymbol{v}\|_{1,2}^2 \leq C(\|\boldsymbol{F}\|_\infty, M). \tag{4.42}$$

This fact together with estimates for temperature ((4.15) and (4.25)) gives us (4.10), which we wanted to prove. □

4.2 Existence for the approximative system

For the sake of simplicity, we perform all the following steps of the proof of Theorem 3.2 considering $\boldsymbol{F} \in L^\infty(\Omega)$ only. All the corresponding calculations could be rewritten using $\boldsymbol{F} \in L^p(\Omega)$ but the proof would be much less transparent. (We can obtain the existence results even in the general case, but the related calculations would be much more complicated and, of course, we would have to use (4.40) instead of (4.41).)

Theorem 4.2 *Let the assumptions of Theorem 3.2 be satisfied. Let $\varepsilon > 0$ and $k > 0$. Then there exists a strong solution $(\rho, \boldsymbol{v}, s)$ to (4.2)–(4.4) such that*

$$\rho \in W^{2,p}(\Omega), \quad \boldsymbol{v} \in W^{2,p}(\Omega) \quad \text{and} \quad s \in W^{2,p}(\Omega) \qquad \text{for } 1 \le p < \infty. \tag{4.43}$$

Moreover $0 \le \rho \le k+1$ in Ω, $\int_\Omega \rho dx \le M$, $\theta > 0$ and

$$\|\boldsymbol{v}\|_{1,q} + \sqrt{\varepsilon}\|\nabla\rho\|_2 + \|\nabla\theta\|_r + \|\theta\|_q \le C(k), \tag{4.44}$$

where $r = 2$ if $m > 2$ and $r = 1 + \delta$, $\delta < 1$ is arbitrary for $m < 2$.

The proof of this theorem will be split into several lemmae.
Let us denote (for $p \in \langle 1, \infty \rangle$)

$$M_p = \{\boldsymbol{u} \in W^{1,p}(\Omega); \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ at } \partial\Omega\}$$

and let us also define operator

$$S : M_p \to W^{2,p}(\Omega) \text{ for } 1 \le p < \infty$$

such that $\rho = S(\boldsymbol{v})$, where ρ solves the following problem:

$$\begin{aligned} \varepsilon\rho - \varepsilon\Delta\rho &= \varepsilon h K(\rho) - \operatorname{div}(K(\rho)\rho\boldsymbol{v}) \qquad \text{in } \Omega \\ \frac{\partial\rho}{\partial\boldsymbol{n}} &= 0 \qquad \text{at } \partial\Omega \end{aligned} \tag{4.45}$$

(see the continuity equation). With this notation we have

Lemma 4.3 *Operator S, defined by (4.45), is a well-defined compact operator from M_p to $W^{2,p}(\Omega)$, $1 \le p < \infty$; the solution to (4.45) is unique. Moreover, for $p > 2$, we have*

$$\|\rho\|_{2,p} \le C(k,\varepsilon)(\|\boldsymbol{v}\|_{1,p}^2 + 1). \tag{4.46}$$

Proof. See [MP], Proposition 3.1, and [NoS], Proposition 4.22; the only difference in our case is the estimates on $\|\rho\|_{2,p}$. We state the idea of the proof here: we take the definition of operator S (4.45) and estimate $\|\nabla\rho\|_p$ from there. The worst term is $\operatorname{div}(K(\rho)\rho\boldsymbol{v})$, so we get

$$\|\nabla\rho\|_p \le C(1 + \|K(\rho)\rho\boldsymbol{v}\|_p) \le C(k)(1 + \|\boldsymbol{v}\|_p)$$

and therefore

$$\|\nabla^2\rho\|_p \le C(1 + \|\nabla\rho\cdot\boldsymbol{v}\|_p + \|\rho\operatorname{div}\boldsymbol{v}\|_p) \le \\ C(1 + \|\boldsymbol{v}\|_\infty\|\nabla\rho\|_p + k\|\operatorname{div}\boldsymbol{v}\|_p) \le C(k)(1 + \|\boldsymbol{v}\|_{1,p}^2)$$

for $p > 2$. □

Now we define another operator, this time with the help of momentum and energy equation:

$$T : M_p \times W^{2,p}(\Omega) \to M_p \times W^{2,p}(\Omega) \quad \text{s.t.} \quad T(\boldsymbol{v}, s) = (\boldsymbol{w}, z),$$

where $(\boldsymbol{w}, z)$ is given as a solution to

$$\begin{aligned} -\operatorname{div}\mathbf{S}(\boldsymbol{w}) &= -\frac{1}{2}\operatorname{div}(K(\rho)\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \frac{1}{2}K(\rho)\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \\ &\qquad \nabla P(\rho, e^s) + K(\rho)\rho\boldsymbol{F} \quad \text{in } \Omega, \\ -\operatorname{div}((1+e^{ms})(\varepsilon + e^s)\nabla z) &= \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - \operatorname{div}\left(\boldsymbol{v}\int_0^\rho K(t)dt\right)e^s - \\ &\qquad \operatorname{div}(K(\rho)\rho\boldsymbol{v})\nabla s - e^sK(\rho)\rho\boldsymbol{v}\cdot\nabla s + e^sK(\rho)\boldsymbol{v}\nabla\rho \quad \text{in } \Omega, \\ \boldsymbol{w}\cdot\boldsymbol{n} = 0, &\qquad \boldsymbol{n}\cdot\mathbf{S}(\boldsymbol{w})\cdot\boldsymbol{\tau} + f\boldsymbol{w}\cdot\boldsymbol{\tau} = 0 \quad \text{at } \partial\Omega, \\ (1+e^{ms})(\varepsilon + e^s)\nabla z + \varepsilon z &= -L(e^s)(e^s - \theta_0) \quad \text{at } \partial\Omega, \end{aligned} \tag{4.47}$$

where $\rho = S(\boldsymbol{v})$ is given by (4.45) and Lemma 4.3.

To apply the Leray–Schauder fixed point theorem we need to verify that T is continuous and compact mapping from $M_p \times W^{2,p}(\Omega)$ to $M_p \times W^{2,p}(\Omega)$ and that all solutions satisfying

$$tT(\boldsymbol{w}, z) = (\boldsymbol{w}, z), \qquad t \in \langle 0, 1\rangle \tag{4.48}$$

are bounded in $M_p \times W^{2,p}(\Omega)$. (We calculate with the fact that $\boldsymbol{w}$ and z are in the right spaces; this fact we prove at the end of this section.)

Lemma 4.4 *Let $p > 2$; let all assumptions of Theorem 4.2 be satisfied. Then T is a continuous and compact operator from $M_p \times W^{2,p}(\Omega)$ to $M_p \times W^{2,p}(\Omega)$.*

Proof. We are going to use two facts: first, $\forall \varepsilon > 0$ system (4.47) is strictly elliptic; second, for $p > 2$ the $W^{1,p}(\Omega)$ space is an algebra. Due to the second fact the RHS of (4.47) belongs to $L^p(\Omega)$ and the boundary terms belong to $W^{1-1/p,p}(\partial\Omega)$.

The coefficients on the LHS of the second equation of (4.47) are of $C^{1+\alpha}(\overline{\Omega})$. The standard elliptic theory gives us information about existence of solution to (4.47) in $M_p \times W^{2,p}(\Omega)$ with the following bound:

$$\|\boldsymbol{w}\|_{2,p} + \|z\|_{2,p} \leq C(\|e^s\|_{C^{1+\alpha}(\Omega)})(\|RHS \text{ of } (4.47)_1\|_p + \|RHS \text{ of } (4.47)_2\|_p + \|RHS \text{ of } (4.47)_4\|_{W^{1-1/p,p}(\partial\Omega)}).$$

This gives us uniqueness of the solution and continuous dependence on the data.

Moreover, the RHS of (4.47) is at most of the first order of sought functions, which implies the compactness of the operator T. □

Lemma 4.5 *All solutions to (4.48) in $M_p \times W^{2,p}(\Omega)$ satisfy the following bounds:*

$$0 \leq \rho \leq k+1, \qquad \|\boldsymbol{w}\|_{1,2} + \|\theta\|_q + \|\nabla\theta\|_{1+\delta} + \sqrt{\varepsilon}\|\nabla\rho\|_2 \leq C(k), \quad (4.49)$$

where $\theta = e^z$, $\delta = 1$ for $m \geq 2$ and $\delta < 1$ for $m < 2$, and $C(k)$ is independent of ε and $t \in \langle 0, 1\rangle$.

Proof. We may copy the estimates from Lemma 4.1; we have to be a bit careful about dependence of all norms on t. On the other hand, we are allowed to use the L^∞ bound on the density, i.e. we can consider $\rho \leq k+1$. So, if we repeat steps (4.11)–(4.17) we get this inequality:

$$(1-t)\int_\Omega \mathbf{S}(\boldsymbol{w}) : \nabla \boldsymbol{w} dx + \int_{\partial\Omega} f(\boldsymbol{w}\odot\boldsymbol{\tau})^2 d\sigma + \int_\Omega \frac{(1+\theta^m)(\varepsilon+\theta)}{\theta}|\nabla z|^2 dx + t\int_\Omega \Big(\frac{\mathbf{S}(\boldsymbol{w}) : \nabla\boldsymbol{w}}{\theta} + \varepsilon\gamma\rho^{\gamma-2}|\nabla\rho|^2 + \frac{\varepsilon\gamma}{\gamma-1}\rho^\gamma\Big)dx + \varepsilon\int_{\partial\Omega}\big(z_+(1-e^{-z_+}) + |z_-|(e^{|z_-|}-1)\big)d\sigma +$$

$$t\int_{\partial\Omega}\big(L(\theta)\theta - L(\theta)\theta_0 + \frac{L(\theta)\theta_0}{\theta} - L(\theta)\big)d\sigma$$
$$\leq t\int_\Omega\Big(K(\rho)\rho\boldsymbol{w}\cdot\nabla z - K(\rho)\boldsymbol{w}\cdot\nabla\rho\Big)dx + tC\Big(1+\int_\Omega|K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}|dx\Big), \tag{4.50}$$

where $\rho = S(\boldsymbol{v})$. Next, we compute again (4.18) – (4.24) and we get

$$\int_\Omega\frac{1+\theta^m}{\theta^2}|\nabla\theta|^2dx + t\int_\Omega\frac{\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}}{\theta}dx + \int_{\partial\Omega}\Big(tL(\theta)\theta + t\frac{L(\theta)\theta_0}{\theta} + \varepsilon|z|\Big)d\sigma$$
$$\leq tC\Big(1+\int_\Omega|K(\rho)\rho\boldsymbol{w}\cdot\boldsymbol{F}|dx\Big). \tag{4.51}$$

We know that $0\leq\rho\leq k+1$, so after dividing by t we get

$$\|\theta\|_q \leq C(1+\|\boldsymbol{w}\|_2)^{1/m}$$
$$\|\boldsymbol{w}\|_{1,2}^2 \leq C(1+\|\theta\|_{1,2}^2).$$

For $m>1$ we get

$$\|\boldsymbol{w}\|_{1,2} + \|\theta\|_q \leq C(k);$$

for $m\geq 2$, from Lemma 4.1 we have L^2 bounds on $\frac{|\nabla\theta|}{\theta}$ and $|\nabla\theta|\theta^{\frac{m-2}{2}}$ and consequently we control also $\nabla\theta$. For $1<m<2$

$$\|\nabla\theta\|_{1+\delta} \leq C(\|\theta\|_q, \||\nabla\theta|\theta^{\frac{m-2}{m}}\|_2)$$

for any $\delta<1$.

To finish the proof we multiply the approximative momentum equation (4.3) by ρ and we integrate by parts to get

$$\varepsilon\int_\Omega(|\nabla\rho^2| + \rho^2)dx \leq \varepsilon\int_\Omega hK(\rho)\rho dx + \int_\Omega\int_0^\rho tK(t)dt|\operatorname{div}\boldsymbol{w}|dx,$$

from where we extract the bound for $\sqrt{\varepsilon}\|\rho\|_2$. □

At last we want to verify the bounds on $\boldsymbol{w}$ and z, i.e. to make sure our operator T maps $M_p\times W^{2,p}(\Omega)$ to $M_p\times W^{2,p}(\Omega)$. We make so by applying the bootstrap method to the system

$$-\operatorname{div}\mathbf{S}(\boldsymbol{w}) = t\Big(-\frac{1}{2}\operatorname{div}(K(\rho)\rho\boldsymbol{w}\otimes\boldsymbol{w}) - \frac{1}{2}K(\rho)\rho\boldsymbol{w}\cdot\nabla\boldsymbol{w} - \nabla P(\rho,e^z) + K(\rho)\rho\boldsymbol{F}\Big) \qquad \text{in } \Omega \tag{4.52}$$

$$-\operatorname{div}\big((1+e^{mz})(\varepsilon+e^z)\nabla z\big) = t\Big(\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w} - \operatorname{div}\Big(\boldsymbol{w}\int_0^\rho K(t)dt\Big)e^z - \operatorname{div}\big(K(\rho)\rho\boldsymbol{w}\big)e^z - e^z K(\rho)\rho\boldsymbol{w}\cdot\nabla z + e^z K(\rho)\boldsymbol{w}\cdot\nabla\rho\Big) \qquad \text{in } \Omega \tag{4.53}$$

with $\rho = S(\boldsymbol{v})$; the boundary conditions are as follows

$$\begin{gathered}\boldsymbol{w}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{n}\cdot\mathbf{S}(\boldsymbol{w})\cdot\boldsymbol{\tau} + f\boldsymbol{w}\cdot\boldsymbol{\tau} = 0 \\ (1+e^{mz})(\varepsilon+e^z)\nabla z + \varepsilon z = -tL(e^z)(e^z-\theta_0).\end{gathered} \tag{4.54}$$

First knowledge (and the one we have for free) is that $\boldsymbol{w}\in W^{1,2}(\Omega)\hookrightarrow L^q(\Omega)$ for any $q\in\langle 1,\infty)$. Thus, when deducing bounds to the RHS of (4.52), the most restrictive term is $\nabla P(\rho, e^z)$ – but we have $\theta\in L^q(\Omega)$ and $\rho\in L^\infty(\Omega)$ and that's why we have also (RHS of (4.52)) $\in (W^{1,q'}(\Omega))^* \Rightarrow \boldsymbol{w}\in W^{1,q}(\Omega)$. As a consequence, from the continuity equation (4.2) we get $\rho\in W^{2,q}(\Omega)$.

Next step is to rewrite equation (4.53):

$$\begin{gathered}-\Delta\Phi(z) = t\Big(\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w} - e^z K(\rho)\rho\boldsymbol{w}\cdot\nabla z + e^z K(\rho)\boldsymbol{w}\cdot\nabla\rho - \operatorname{div}\Big(\boldsymbol{w}\int_0^\rho K(t)dt\Big)e^z - \operatorname{div}\big(K(\rho)\rho\boldsymbol{w}\big)e^z\Big) \qquad \text{in } \Omega, \\ \frac{\partial\Phi(z)}{\partial\boldsymbol{n}} = -\varepsilon z - tL(e^z)(e^z-\theta_0) \qquad \text{at } \partial\Omega,\end{gathered} \tag{4.55}$$

where

$$\Phi(z) = \int_0^z (1+e^{m\tau})(\varepsilon+e^\tau)d\tau.$$

We need to verify that Φ is bounded. We multiply (4.52) by Φ and integrate over Ω. It leads to

$$\|\nabla\Phi\|_2^2 + \int_{\partial\Omega}\big(tL(e^z)(e^z-\theta_0)\Phi + \varepsilon z\Phi\big)\,d\sigma \le C\|\text{RHS of } (4.52)\|_{\frac{q}{q-1}}\|\Phi\|_q.$$

Now we realize that $\Phi(s)\sim\varepsilon s$ for $s\to-\infty$ and $\Phi(s)\sim e^{(m+1)s}$ for $s\to\infty$; from this fact we deduce that

$$\begin{gathered}\int_{\partial\Omega}(tL(e^s)(\theta-\theta_0)\Phi + \varepsilon s\Phi)I_{\{\Phi\le 0\}}d\sigma \ge \|\Phi\|_2^2 - C \\ \int_{\partial\Omega}(tL(e^s)(\theta-\theta_0)\Phi + \varepsilon s\Phi)I_{\{\Phi\ge 0\}}d\sigma \ge \|\Phi\|_1 - C\end{gathered}$$

and therefore $\|\Phi\|_{1,2} \le C$ with C independent of t. This fact implies

$$\nabla\theta = e^z \nabla z \in L^2(\Omega).$$

Consequently we have (using imbedding theorems) $\Phi \in W^{1,\widetilde{q}}(\Omega)$ for arbitrary $\widetilde{q}$. Now, we are ready to show that $\Phi \in W^{2,\widetilde{q}}(\Omega)$ (the method is just the same as in previous case, we use standard elliptic theory) and from this fact we conclude that

$$z \in W^{2,\widetilde{q}}(\Omega) \hookrightarrow L^\infty(\Omega), \qquad \nabla z \in W^{1,\widetilde{q}}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Using all these facts and equation (4.52) we get also

$$\boldsymbol{w} \in W^{2,q}(\Omega).$$

This finishes the proof of Theorem 4.2, because now we have

$$\|\boldsymbol{w}\|_{2,r} + \|z\|_{2,r} + \|\theta\|_{2,r} \le C, \qquad 1 \le r < \infty,$$

where the constant C does not depend on t. Moreover, $\theta = e^z$, $z \in L^\infty$, so we have $\theta \ge c(\varepsilon) > 0$.

5 Proof II: Convergence

From estimates from Theorem 4.2 we know that there exists a subsequence $\varepsilon \to 0+$ such that:

$$
\begin{gathered}
\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v} \qquad \text{in } W^{1,q}(\Omega), \\
\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v} \qquad \text{in } L^\infty(\Omega), \\
\rho_\varepsilon \rightharpoonup^* \rho \qquad \text{in } L^\infty(\Omega), \\
P_b(\rho_\varepsilon) \rightharpoonup^* \overline{P_b(\rho)} \qquad \text{in } L^\infty(\Omega), \\
K(\rho_\varepsilon)\rho_\varepsilon \rightharpoonup^* \overline{K(\rho)\rho} \qquad \text{in } L^\infty(\Omega), \\
K(\rho_\varepsilon) \rightharpoonup^* \overline{K(\rho)} \qquad \text{in } L^\infty(\Omega), \\
\textstyle\int_0^{\rho_\varepsilon} K(t)dt \rightharpoonup^* \overline{\int_0^{\rho} K(t)dt} \qquad \text{in } L^\infty(\Omega), \\
\theta_\varepsilon \rightharpoonup \theta \qquad \text{in } W^{1,1+\delta}(\Omega), \delta < 1 \text{ arbitrary}, \\
\theta_\varepsilon \to \theta \qquad \text{in } L^q(\Omega),
\end{gathered}
\tag{5.1}
$$

where the bar over a quantity denotes its weak limit for $\varepsilon \to 0+$.

(The first line is due to Theorem 2.8, the second we have from imbedding theorems for Sobolev spaces (see Theorem 2.2). Third line is application of Theorem 2.9. The last two lines are again due to Theorem 2.8. The four lines between are a simple consequence of properties of weak-* limits.)

With this knowledge the limit of our problem looks a bit more friendly:

$$
\operatorname{div}(\overline{K(\rho)\rho}\boldsymbol{v}) = 0 \tag{5.2}
$$

$$
\begin{aligned}
-\operatorname{div}\Big(2\mu\mathbf{D}(\boldsymbol{v}) + \nu(\operatorname{div}\boldsymbol{v})\mathbf{I} - \overline{P_b(\rho)}\mathbf{I} - \theta\big(\overline{\int_0^\rho K(t)dt}\big)\mathbf{I}\Big) + \\
\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \overline{K(\rho)\rho}\boldsymbol{F}
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
\operatorname{div}((1+\theta^m)\nabla\theta) + \theta\big(\overline{\operatorname{div}\boldsymbol{v}\int_0^\rho K(t)dt}\big) + \operatorname{div}(\overline{K(\rho)\rho}\theta\boldsymbol{v}) = \\
2\mu\overline{|\mathbf{D}(\boldsymbol{v})|^2} + \nu\overline{(\operatorname{div}\boldsymbol{v})^2}
\end{aligned}
\tag{5.4}
$$

together with the boundary conditions (1.10) and (1.11); the fact that the boundary conditions are fulfilled is not trivial and we will give further comments on it at the end of this section.

Note that in the equation (5.3), especially in the last term on the LHS, we used the fact that $\operatorname{div}(\overline{K(\rho)\rho}\boldsymbol{v}) = 0$.

Lemma 5.1 *Under the assumptions of Theorem 3.2 and Theorem 4.2, we have*

$$\|\rho_\varepsilon\|_\infty \leq k+1 \qquad \text{and} \qquad \|\boldsymbol{v}_\varepsilon\|_{1,q} \leq C(1 + k^{1+\frac{1}{q+\delta}} + k^{\gamma\frac{q-2}{q}}), \tag{5.5}$$

where $\delta < 1$ *arbitrarily small.*

Proof. The bound on the density follows directly from Theorem 4.2 – therefore we are going to estimate the velocity. If we write the approximative momentum equation (4.3) in the form

$$-\operatorname{div}\mathbf{S}(\boldsymbol{v}) = -\nabla\Bigg(P_b(\rho_\varepsilon) + \theta_\varepsilon(\int_0^{\rho_\varepsilon} K(t)dt)\Bigg) + K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{F} - \frac{1}{2}\operatorname{div}\big(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon\big) - \frac{1}{2}K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon,$$

we can notice that

$$\|\boldsymbol{v}_\varepsilon\|_{1,q} \leq C\Bigg(\|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon\|_q + \|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon\|_{\frac{2q}{q+2}} + \|P_b(\rho_\varepsilon)\|_q + \|\theta_\varepsilon(\int_0^{\rho_\varepsilon} K(t)dt)\|_q + \|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{F}\|_{\frac{2q}{q+2}}\Bigg). \tag{5.6}$$

The bounds on the density and temperature yield

$$\|P_b(\rho_\varepsilon)\|_q \leq \|P_b(\rho_\varepsilon)\|_2^{\frac{2}{q}}\|P_b(\rho_\varepsilon)\|_\infty^{\frac{q-2}{q}} \leq Ck^{\gamma\frac{q-2}{q}} \tag{5.7}$$

and

$$\|\theta_\varepsilon(\int_0^{\rho_\varepsilon} K(t)dt)\|_q \leq C\|\rho\|_{q+\delta}\|\theta\|_{p(q,\delta)} \leq Ck^{1+\frac{1}{q+\delta}}. \tag{5.8}$$

(All these inequalities work for sufficiently large q and k.) The only thing that remains is to estimate the convective term:

$$\|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon\|_q + \|K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon\|_{\frac{2q}{q+2}} \leq C\|\boldsymbol{v}\|_{1,2}^2\|P_b(\rho)\|_{\frac{q}{\gamma}}^{\frac{1}{\gamma}},$$

from where we directly deduce (using equations (5.6), (5.7) and (5.8) and also the fact that $\gamma > 2$) the main bound

$$\|\boldsymbol{v}_\varepsilon\|_{1,q} \leq C(k^{1+\frac{1}{q+\delta}} + k^{\gamma\frac{q-2}{q}}).$$

□

5.1 Convergence of the temperature

Lemma 5.2 *There exists a subsequence* $\{s_\varepsilon\}$ *such that*

$$s_\varepsilon \to s \qquad \text{in } L^2(\Omega),$$

consequently

$$\theta_\varepsilon \to \theta \qquad \text{in } L^q(\Omega) \text{ for } q < \infty,$$

where $\theta > 0$ *a.e. in* Ω.

Proof. From the previous sections we have

$$\int_\Omega |\nabla s_\varepsilon|^2 dx + \int_{\partial\Omega} (e^{s_\varepsilon} + e^{-s_\varepsilon}) d\sigma < C$$

and especially

$$\int_\Omega |\nabla s_\varepsilon|^2 dx + \int_{\partial\Omega} |s_\varepsilon|^2 d\sigma < C.$$

By the "definition" of equivalent norms in Sobolev spaces (see (2.4)) and by the imbedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ we know that there exists subsequence $s_\varepsilon \to s$ in $L^2(\Omega)$. Now we remember the fact that $\theta_\varepsilon = e^{s_\varepsilon}$ and $\theta_\varepsilon \to \theta$ in $L^q(\Omega)$, see the last line of (5.1). Now we use Vitali's theorem (Theorem 2.10) to get

$$e^{s_\varepsilon} \to e^s \qquad \text{in } L^q(\Omega)$$

and

$$\theta = e^s \quad \text{with } s \in L^2(\Omega).$$

Thus $\theta > 0$ a.e. in Ω as $s > -\infty$ a.e. in Ω. □

5.2 Effective viscous flux

To prove the strong convergence of the density we need an interesting quantity called effective viscous flux. To define it, we have to work with Helmholtz decomposition of the velocity for a while:

$$\boldsymbol{v}_\varepsilon = \nabla^\perp A_\varepsilon + \nabla \phi_\varepsilon, \tag{5.9}$$

where $\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$, the scalar function ϕ_ε is given by the system

$$\begin{aligned} \Delta \phi_\varepsilon &= \operatorname{div} \boldsymbol{v}_\varepsilon \qquad \text{in } \Omega \\ \frac{\partial \phi_\varepsilon}{\partial \boldsymbol{n}} &= 0 \qquad \text{at } \partial\Omega \end{aligned} \tag{5.10}$$

and the field A_ε is given by

$$\begin{aligned} \Delta A_\varepsilon = \operatorname{rot} \boldsymbol{v}_\varepsilon = \omega_\varepsilon \qquad & \text{in } \Omega \\ \boldsymbol{n} \cdot \nabla^\perp A_\varepsilon = 0 \qquad & \text{at } \partial\Omega, \end{aligned} \tag{5.11}$$

where $\operatorname{rot} \boldsymbol{v}_\varepsilon = \frac{\partial v_{\varepsilon 2}}{\partial x_1} - \frac{\partial v_{\varepsilon 1}}{\partial x_2}$ (as we consider the two-dimensional case; note that in 2D $\operatorname{rot} \boldsymbol{v}_\varepsilon$ is scalar function). The basic estimates (following from the the standard elliptic estimates) for A_ε and ϕ are

$$\begin{aligned} \|\nabla\nabla^\perp A_\varepsilon\|_q \le C\|\omega_\varepsilon\|_q \qquad & \|\nabla^2\nabla^\perp A_\varepsilon\|_q \le C\|\omega_\varepsilon\|_{1,q} \\ \|\nabla^2\phi_\varepsilon\|_q \le C\|\operatorname{div} \boldsymbol{v}_\varepsilon\|_q \qquad & \|\nabla^3\phi_\varepsilon\|_q \le C\|\operatorname{div} \boldsymbol{v}_\varepsilon\|_{1,q}. \end{aligned} \tag{5.12}$$

We have (in a weak sense)

$$\begin{aligned} -\mu\Delta\omega_\varepsilon = -\frac{1}{2}\operatorname{rot}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon) - \frac{1}{2}\operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon) + & \\ \operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{F}) \qquad & \text{in } \Omega \\ \omega_\varepsilon = \left(2\chi - \frac{f}{\mu}\right) \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\tau} \qquad & \text{at } \partial\Omega, \end{aligned} \tag{5.13}$$

where χ is the curvature of $\partial\Omega$, cf. [Mu].

The form of system (5.13) enables to state the following two problems for $\omega_\varepsilon = \omega_\varepsilon^1 + \omega_\varepsilon^2$:

$$\begin{aligned} -\mu\Delta\omega_\varepsilon^1 = -\frac{1}{2}\operatorname{rot}\left(\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v})\boldsymbol{v}\right) \qquad & \text{in } \Omega \\ \omega_\varepsilon^1 = 0 \qquad & \text{at } \partial\Omega; \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} -\mu\Delta\omega_\varepsilon^2 = -\operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon) + \operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{F}) \qquad & \text{in } \Omega \\ \omega_\varepsilon^2 = \left(2\chi - \frac{f}{\mu}\right) \boldsymbol{v}_\varepsilon \cdot \boldsymbol{\tau} \qquad & \text{at } \partial\Omega. \end{aligned} \tag{5.15}$$

From these equations we get

$$\|\omega_\varepsilon^1\|_q \le C\|K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon\|_q \le \sqrt{\varepsilon}C(k) \tag{5.16}$$

$$\|\omega_\varepsilon^2\|_q \le C(1 + \|\nabla \boldsymbol{v}_\varepsilon\|_2^2 + \|K(\rho_\varepsilon)\rho_\varepsilon\|_{\frac{q}{q-2}}) \le Ck^{\gamma\frac{q-2}{q}}. \tag{5.17}$$

At the first relation we used the approximative continuity equation (4.2):

$$\operatorname{rot}(\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon) = \operatorname{rot}(\boldsymbol{v}_\varepsilon \varepsilon h + \varepsilon\Delta\rho_\varepsilon + \varepsilon\rho_\varepsilon)$$

and when estimating the term with $\Delta\rho_\varepsilon$, we use the fact that $\sqrt{\varepsilon}\|\nabla\rho_\varepsilon\| < C$, which imlpies also the presence of the square root in the estimate.

At last we are ready to introduce the fundamental quantity which is in fact the potential part of the momentum equation: the effective viscous flux. Using the Helmholtz decomposition in the approximative momentum equation we have

$$\begin{aligned}\nabla(-(2\mu+\nu)\Delta\phi_\varepsilon + P(\rho_\varepsilon,\theta_\varepsilon)) = \mu\Delta\nabla^\perp A_\varepsilon + K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{F}\\ -K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon hK(\rho_\varepsilon)\boldsymbol{v}_\varepsilon + \frac{1}{2}\varepsilon\rho_\varepsilon\boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon\Delta\rho_\varepsilon\boldsymbol{v}_\varepsilon.\end{aligned} \tag{5.18}$$

We define

$$G_\varepsilon = -(2\mu+\nu)\Delta\phi_\varepsilon + P(\rho_\varepsilon,\theta_\varepsilon) = -(2\mu+\nu)\operatorname{div}\boldsymbol{v}_\varepsilon + P(\rho_\varepsilon,\theta_\varepsilon) \tag{5.19}$$

and its limit version

$$G = -(2\mu+\nu)\operatorname{div}\boldsymbol{v} + \overline{P(\rho,\theta)}. \tag{5.20}$$

In the following lemmae we prove fundamental properties of the effective viscous flux.

Lemma 5.3

$$\|G\|_\infty \le C(1+k^{\frac{3}{2}+\eta}) \text{ with } \eta>0 \text{ arbitrarily small}, \qquad \|G\|_2 \le C. \tag{5.21}$$

Proof. We pass to the limit in (5.18), getting

$$\nabla((-2\mu+\nu)\Delta\phi + \overline{P(\rho,\theta)}) = \mu\Delta\nabla^\perp A + \overline{K(\rho)\rho}\boldsymbol{F} - \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v};$$

from this equation we estimate $\|\nabla G\|_q$ $(q>2)$:

$$\|\nabla G\|_q \le C(\|\Delta\nabla^\perp A\|_q + \|\overline{K(\rho)\rho}\boldsymbol{F}\|_q + \|\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q).$$

We still remember the results from Lemma 5.1 and so

$$\|K(\rho)\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q \le C\|\nabla\boldsymbol{v}\|_q^2\|\overline{K(\rho)\rho}\|_\infty \le Ck^{\frac{3}{2}+\eta},$$

with η arbitrarily small for $q\to 0$ (the second term from the second inequality form (5.5) is very small for $q\to 0$). Finally, we have in the weak sense

$$\begin{aligned}-\mu\Delta\omega = -\operatorname{rot}(\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}) - \operatorname{rot}(\overline{K(\rho)\rho}\boldsymbol{F}) \qquad &\text{in } \Omega\\ \omega = \left(2\chi - \frac{f}{\mu}\right)\boldsymbol{v}\cdot\boldsymbol{\tau} \qquad &\text{at } \partial\Omega,\end{aligned}$$

thus

$$\|\omega\|_{1,q} \le C(\|\boldsymbol{v}\|_{1,q} + \|\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q + \|\overline{K(\rho)\rho}\boldsymbol{F}\|_q) \le Ck^{\frac{3}{2}+\eta}.$$

We also have (see (5.12))

$$\|\Delta\nabla^\perp A\|_q \le C\|\nabla\omega\|_q;$$

all this, together with control of mean value of G (in fact, mean value of G is mean value of $\overline{P(\rho,\theta)}$) and Sobolev imbeddings finishes the proof of the first inequality. The second inequality we get quickly due to the fact that

$$\|G\|_2 \le C(\|\nabla\boldsymbol{v}\|_2 + \|P(\rho,\theta)\|_2).$$

□

Lemma 5.4 *We have (up to a subsequence $\varepsilon \to 0+$):*

$$G_\varepsilon \to G \text{ strongly in } L^2(\Omega) \tag{5.22}$$

Proof.

$$\begin{aligned}\nabla(G_\varepsilon - G) = (K(\rho_\varepsilon)\rho_\varepsilon - \overline{K(\rho)\rho})\boldsymbol{F} - \frac{1}{2}K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \\ \frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) + \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \mu\Delta\nabla^\perp(A_\varepsilon - A).\end{aligned} \tag{5.23}$$

For the first term we have

$$(K(\rho_\varepsilon)\rho_\varepsilon - \overline{K(\rho)\rho})\boldsymbol{F} \rightharpoonup 0 \qquad \text{in } L^q(\Omega)\ \forall q < \infty;$$

we know that for a general sequence

$$\nabla(f_\varepsilon - f) \rightharpoonup 0 \quad \text{in } L^q(\Omega) \quad \Rightarrow \quad f_\varepsilon - f \to \text{const} \quad \text{in } L^q(\Omega),$$

so the first term gives us strong convergence. The second “part” is

$$\begin{aligned}\frac{1}{2}K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) + \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \\ \frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon + K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v},\end{aligned}$$

and for the first term it holds

$$\frac{1}{2}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon = \frac{1}{2}\varepsilon\Delta\rho_\varepsilon \boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon\rho_\varepsilon \boldsymbol{v}_\varepsilon + \frac{1}{2}\varepsilon h K(\rho)\boldsymbol{v}_\varepsilon,$$

in which the first term converges to zero strongly in $W^{-1,2}$ (this determines the space of convergence) and the other two terms converge to zero weakly in $L^q(\Omega)$, $q<\infty$ (the same argument as above plus the fact that $K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon\cdot\nabla \boldsymbol{v}_\varepsilon$ is bounded in any $L^q(\Omega)$).

At last we have to look on the last term of (5.23). We show that

$$\nabla(\omega_\varepsilon - \omega) = B_\varepsilon^1 + B_\varepsilon^2,$$

where $B_\varepsilon^1 \rightharpoonup 0$ in $L^2(\Omega)$ and $B_\varepsilon^2 \to 0$ in $W^{-1,2}(\Omega)$. We have

$$\begin{aligned} \Delta(\omega_\varepsilon - \omega) = -\frac{1}{2}\operatorname{rot}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon\cdot\nabla \boldsymbol{v}_\varepsilon) - \frac{1}{2}\operatorname{rot}\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon\otimes \boldsymbol{v}_\varepsilon)+ \\ \operatorname{rot}(\overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \operatorname{rot}((K(\rho_\varepsilon)\rho_\varepsilon - \overline{K(\rho)\rho})\boldsymbol{F}) \qquad \text{in } L^2(\Omega) \\ \omega_\varepsilon - \omega = \left(2\chi - \frac{f}{\mu}\right)(\boldsymbol{v}_\varepsilon - \boldsymbol{v})\cdot\tau \qquad \text{at } \partial\Omega. \end{aligned}$$

Now, as above,

$$K(\rho_\varepsilon)\rho_\varepsilon \boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \overline{K(\rho)\rho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} = B_\varepsilon^1 + B_\varepsilon^2$$

where $B_\varepsilon^1 \to 0$ strongly in $W^{-1,2}(\Omega)$ and $B_\varepsilon^2 \rightharpoonup 0$ weakly in $L^2(\Omega)$ (the first term makes no trouble, the second one we again rewrite with a help of the approximative continuity equation); the divergence for the rest of the terms is just the same as above. Now, we recall that

$$\|\Delta\nabla^\perp(A_\varepsilon - A)\|_{-1,2} \le \|\nabla(\omega_\varepsilon-\omega)\|_{-1,2}$$

and from that we conclude that

$$\Delta\nabla^\perp(A_\varepsilon - A) = B_\varepsilon^1 + B_\varepsilon^2,$$

where $B_\varepsilon^1 \to 0$ in $W^{-1,2}(\Omega)$ and $B_\varepsilon^2 \rightharpoonup 0$ in $L^2(\Omega)$. Therefore, we get $\nabla(G_\varepsilon - G) \Rightarrow G_\varepsilon - G \to$ const. in $L^2(\Omega)$. But we notice that

$$\int_\Omega (G_\varepsilon - G)dx = \int_\Omega \Delta(\phi_\varepsilon - \phi) + \int_\Omega (P(\rho_\varepsilon,\theta_\varepsilon) - P(\rho,\theta)) \to 0$$

because

$$\int_{\partial\Omega}\frac{\partial\phi}{\partial\boldsymbol{n}}dS = \int_{\partial\Omega}\frac{\partial\phi_\varepsilon}{\partial\boldsymbol{n}}dS = 0$$

and the constant is zero. □

5.3 Limit passage

Theorem 5.5 *There exists sufficiently large $k_0 > 0$ such that for $k > k_0$ we have*

$$\frac{k-3}{k}(k-3)^\gamma \geq 1 + \|G\|_\infty \tag{5.24}$$

and for a subsequence $\varepsilon \to 0+$ it holds

$$\lim_{\varepsilon \to 0+} |\{x \in \Omega : \rho_\varepsilon(x) > k-3\}| = 0. \tag{5.25}$$

In particular, $\overline{K(\rho)\rho} = \rho$ a.e. in Ω.

Proof. We define a smooth function $M : \mathbb{R}_0^+ \to \langle 0, 1\rangle$ such that

$$M(t) = \begin{cases} 1 & \text{for} \quad t \leq k-3 \\ \in \langle 0,1\rangle & \text{for} \quad k-3 < t < k-2 \\ 0 & \text{for} \quad k-2 \leq t \end{cases}$$

and $M'(t) < 0$ for $t \in (k-3, k-2)$. Now, we multiply the approximative continuity equation (4.2) by M^l ($l \in \mathbb{N}$) and integrate over Ω. As

$$\varepsilon \int_\Omega M^l(\rho_\varepsilon)\Delta\rho_\varepsilon dx = -\varepsilon l \int_\Omega M^{l-1}(\rho_\varepsilon)M'(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2 dx \geq 0,$$

we get

$$\int_\Omega \left(\int_0^{\rho_\varepsilon(x)} tlM^{l-1}(t)M'(t)dt \right) div\boldsymbol{v}_\varepsilon \geq R_\varepsilon$$

with $R_\varepsilon \to 0$ as $\varepsilon \to 0$. Now, we recall definitions of G and M to get

$$\begin{aligned} -(k-3)\int_\Omega \Big(\int_0^{\rho_\varepsilon(x)} lM^{l-1}(t)M'(t)dt \Big) P(\rho_\varepsilon, \theta_\varepsilon)dx \\ \leq k\Big| \int_\Omega \Big(\int_0^{\rho_\varepsilon(x)} -lM^{l-1}(t)M'(t)dt \Big) G_\varepsilon dx \Big| + R_\varepsilon. \end{aligned}$$

Thus the properties of M lead us to the following inequality

$$\frac{k-3}{k}\int_{\{\rho_\varepsilon > k-3\}} (1 - M^l(\rho_\varepsilon))P(\rho_\varepsilon, \theta_\varepsilon)dx \leq \int_{\{\rho_\varepsilon > k-3\}} (1 - M^l(\rho_\varepsilon))|G_\varepsilon|dx + |R_\varepsilon|.$$

From the explicit form of approximative pressure function (4.5) we see

$$\frac{k-3}{k}(k-3)^{\gamma}|\{\rho_\varepsilon > k-3\}| - \frac{k-3}{k}\|P(\rho_\varepsilon,\theta_\varepsilon)\|_2\|M^l(\rho_\varepsilon)\|_2 \\ \leq \|G\|_\infty|\{\rho_\varepsilon > k-3\}| + \|G-G_\varepsilon\|_1 + |R_\varepsilon|.$$

By inequality (5.21) (see Lemma 5.4) we are able to choose k_0 so large that for all $k > k_0$ we have (5.24), $\|G\|_\infty \leq Ck^{\frac{3}{2}+\eta}$ and $\gamma > 2$.

Therefore,

$$|\{x \in \Omega : \rho_\varepsilon(x) > k-3\}| \leq C\left(\|M^l(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > k-3\})} + \|G-G_\varepsilon\|_{L^1(\Omega)} + |R_\varepsilon|\right). \tag{5.26}$$

Now, for fixed $\delta > 0$ there exists $\varepsilon_0 > 0$ such that fo $\varepsilon < \varepsilon_0$

$$C(\|G-G_\varepsilon\|_1 + |R_\varepsilon|) \leq \frac{\delta}{2}. \tag{5.27}$$

We fix ε and then consider the sequence $\{M^l(\rho_\varepsilon)I_{\{\rho_\varepsilon > k-3\}}\}_{l\in\mathbb{N}}$, where I_A is the characteristic function of a set A. We see that it monotonely pointwise converges to zero. Thus by the Lebesgue theorem we are able to find $l = l(\varepsilon,\delta)$ such that

$$C\|M^l(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > k-3\})} \leq \frac{\delta}{2}. \tag{5.28}$$

From (5.26), (5.27) and (5.28) we obtain

$$\lim_{\varepsilon\to 0}|\{x \in \Omega; \rho_\varepsilon(x) > k-3\}| \leq \delta. \tag{5.29}$$

As $\delta > 0$ is arbitrarily small, Theorem 5.5 is proved. □

Now we are finally going to prove pointwise convergence of the density.

Lemma 5.6 *We have*

$$\int_\Omega \overline{P(\rho,\theta)\rho}dx \leq \int_\Omega G\rho dx \tag{5.30}$$

and

$$\int_\Omega \overline{P(\rho,\theta)}\rho dx = \int_\Omega G\rho dx; \tag{5.31}$$

$\overline{P(\rho,\theta)\rho} = \overline{P(\rho,\theta)}\rho$ *and up to a subsequence* $\varepsilon \to 0+$

$$\rho_\varepsilon \to \rho \quad \textit{strongly in } L^q(\Omega) \quad \textit{for any } q < \infty. \tag{5.32}$$

Proof. Due to Theorem 5.5 we are able to omit $K(\rho)$ from the limit equation.

We test the approximative continuity equation (4.2) by $\ln(k+1)-\ln(\rho_\varepsilon+\delta)$ for $\delta>0$. It holds

$$\int_\Omega \varepsilon\Delta\rho_\varepsilon(\ln(k+1)-\ln(\rho_\varepsilon+\delta))dx=\varepsilon\int_\Omega|\nabla\rho_\varepsilon|^2\frac{1}{\rho_\varepsilon+\delta}dx\geq 0,$$

and thus

$$\int_\Omega\left(\operatorname{div}(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon)+\varepsilon\rho_\varepsilon-\varepsilon hK(\rho_\varepsilon)\right)\left(\ln(k+1)-\ln(\rho_\varepsilon+\delta)\right)dx\geq 0;$$

the meaning of this is

$$\int_\Omega K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon+\delta}dx\geq\int_\Omega\left(\varepsilon hK(\rho_\varepsilon)-\varepsilon\rho_\varepsilon\right)\left(\ln(k+1)-\ln(\rho_\varepsilon+\delta)\right)dx.$$

The next step is to pass with $\delta\to 0+$. During this operation, the only trouble-making term is the first one on the RHS – we have to realize this:

$$0\leq\int_\Omega\varepsilon hK(\rho_\varepsilon)\ln\frac{k+1}{\rho_\varepsilon+\delta}1_{\{\rho_\varepsilon<k+\frac{1}{2}\}}dx\leq$$
$$\int_\Omega\left(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon+\delta}+\varepsilon\rho_\varepsilon\ln\frac{k+1}{\rho_\varepsilon+\delta}-\varepsilon hK(\rho_\varepsilon)\ln\frac{k+1}{\rho_\varepsilon+\delta}1_{\{\rho_\varepsilon\geq k+\frac{1}{2}\}}\right)dx,$$

where 1_A denotes the characteristic function of a set A, and the Fatou lemma then implies

$$\int_\Omega K(\rho_\varepsilon)\boldsymbol{v}_\varepsilon\cdot\nabla\rho_\varepsilon dx\geq\int_\Omega(1-K(\rho_\varepsilon))\boldsymbol{v}_\varepsilon\cdot\nabla\rho_\varepsilon dx+\varepsilon\int_\Omega hK(\rho_\varepsilon)\ln\frac{k+1}{\rho_\varepsilon}dx-$$
$$\varepsilon\int_\Omega\rho_\varepsilon\ln\frac{k+1}{\rho_\varepsilon}dx\geq-\int_\Omega\left(\int_0^{\rho_\varepsilon}(1-K(t))dt\right)\operatorname{div}\boldsymbol{v}_\varepsilon dx-\varepsilon\int_\Omega\rho_\varepsilon\ln\frac{k+1}{\rho_\varepsilon}dx$$

because the middle term is nonnegative. This finally implies (as $\operatorname{div}\boldsymbol{v}_\varepsilon$ is bounded in L^2 and $|\{x\in\Omega:\rho(x)>k\}|\to 0$)

$$-\int_\Omega\rho_\varepsilon\operatorname{div}\boldsymbol{v}_\varepsilon\geq R_\varepsilon,$$

where $R_\varepsilon\to 0$ as $\varepsilon\to 0$.

From the definition of G_ε we have

$$\int_\Omega P(\rho_\varepsilon,\theta_\varepsilon)\rho_\varepsilon dx+R_\varepsilon\leq\int_\Omega G_\varepsilon\rho_\varepsilon dx$$

and now we recall that $R_\varepsilon \to 0$ for $\varepsilon \to 0$, so we get

$$\int_\Omega \overline{P(\rho,\theta)\rho}dx \leq \int_\Omega G\rho dx.$$

Next, we consider the limit to the continuity equation:

$$\operatorname{div}(\rho \boldsymbol{v}) = 0.$$

We use Lemma 2.11 (its assumptions are fulfilled) to find ρ_n and to see, that

$$\int_\Omega \operatorname{div}(\rho_n \boldsymbol{v})dx = \int_\Omega \rho_n \boldsymbol{v} \cdot \boldsymbol{n} dS = 0.$$

Thus

$$\int_\Omega (\rho_n \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \rho_n)dx = 0$$

and passing with $n \to \infty$ we get

$$\int_\Omega (\rho \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \rho)dx = 0.$$

ρ_n is constructed using partition of the unity and the Friedrich's lemma; we have $\rho_n \in \langle 0, k+1 \rangle$. We take $\delta > 0$ and test the continuity equation by $-\ln \frac{\delta}{\rho_n + \delta}$ and we get

$$0 = -\int_\Omega \rho \boldsymbol{v} \cdot \nabla \ln \frac{\delta}{\delta + \rho_n} dx = \int_\Omega \frac{\rho \boldsymbol{v} \cdot \nabla \rho_n}{\rho_n + \delta} dx$$

and passing with $n \to \infty$ we get

$$0 = \int_\Omega \frac{\rho}{\rho + \delta} \boldsymbol{v} \cdot \nabla \rho dx.$$

Finally, passing with $\delta \to 0$ we get

$$\int_\Omega \rho \operatorname{div} \boldsymbol{v} dx = 0,$$

and from the definition of G we get

$$\int_\Omega \overline{P(\rho,\theta)}\rho dx = \int_\Omega G\rho dx.$$

The elementary properties of weak limits gives us $\rho\overline{P(\rho,\theta)} \le \overline{P(\rho,\theta)\rho}$ a.e. in Ω, but (5.30) and (5.31) implies $\int_\Omega(\overline{P(\rho,\theta)\rho} - \overline{P(\rho,\theta)}\rho)dx \le 0$, hence

$$\rho\overline{P(\rho,\theta)} = \overline{P(\rho,\theta)\rho} \quad \text{a.e.,} \qquad \text{i.e.} \quad \overline{\rho^{\gamma+1}} + \overline{\rho^2}\theta = \overline{\rho^\gamma}\rho + \rho^2\theta \quad \text{a.e.}$$

The same elementary properties tell us that $\overline{\rho^{\gamma+1}} \ge \overline{\rho^\gamma}\rho$ and $\overline{\rho^2}\theta \ge \rho^2\theta$, so

$$\overline{\rho^{\gamma+1}} = \overline{\rho^\gamma}\rho \quad \text{a.e.} \qquad \text{and} \qquad \overline{\rho^2}\theta = \rho^2\theta \quad \text{a.e.}$$

From Lemma 5.2 we know that $\theta > 0$ a.e. and thus $\overline{\rho^2} = \rho^2$ and for a subsequence

$$\lim_{\varepsilon\to 0} \|\rho_\varepsilon - \rho\|_2^2 = \overline{\rho^2} - \rho^2 = 0. \tag{5.33}$$

□

The only detail that remains to solve before finishing the proof of Theorem 3.2 is the rest of the convergencies. First, due to Theorem 5.5 and Lemma 5.2 we have

$$P(\rho_\varepsilon,\theta_\varepsilon) \to p(\rho,\theta) \qquad \text{strongly in } L^2(\Omega).$$

From this equation and from (5.22) we deduce that

$$\operatorname{div} \boldsymbol{v}_\varepsilon \to \operatorname{div} \boldsymbol{v} \qquad \text{strongly in } L^2(\Omega). \tag{5.34}$$

Additionaly, from the properties of the vorticity we already know that

$$\operatorname{rot} \boldsymbol{v}_\varepsilon \to \operatorname{rot} \boldsymbol{v} \qquad \text{strongly in } L^2(\Omega). \tag{5.35}$$

All this together with the regularity of (5.10) and (5.11) gives us

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \qquad \text{strongly in } W^{1,2}(\Omega). \tag{5.36}$$

In particular

$$\mathbf{S}(\boldsymbol{v}_\varepsilon) : \nabla\boldsymbol{v}_\varepsilon \to \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} \qquad \text{strongly at least in } L^1(\Omega). \tag{5.37}$$

Now, it is time for a little summary. We know, that

$$\begin{aligned}
&\rho_\varepsilon \to \rho && \text{in } L^q(\Omega) \text{ for } q < \infty \\
&\boldsymbol{v}_\varepsilon \to \boldsymbol{v} && \text{in } L^q(\Omega) \text{ for } q < \infty \\
&\theta_\varepsilon \to \theta && \text{in } L^q(\Omega) \text{ for } q < \infty \\
&\theta_\varepsilon \rightharpoonup \theta && \text{in } W^{1,2}(\Omega) \text{ if } m > 2 \\
&\theta_\varepsilon \rightharpoonup \theta && \text{in } W^{1,1+\delta}(\Omega) \text{ for } \delta < 1 \text{ if } m > 2.
\end{aligned} \tag{5.38}$$

We return to the approximative energy equation (4.4):

$$\begin{aligned}\int_\Omega (1+\theta_\varepsilon^m)\frac{\varepsilon+\theta_\varepsilon}{\theta_\varepsilon}\nabla\theta_\varepsilon\cdot\nabla\phi dx + \int_{\partial\Omega} L(\theta_\varepsilon)(\theta_\varepsilon-\theta_0)\phi d\sigma + \int_{\partial\Omega}\varepsilon\ln\theta_\varepsilon\phi d\sigma - \\ \int_\Omega \left(\left(\int_0^{\rho_\varepsilon(x)} K(t)dt\right)\boldsymbol{v}_\varepsilon\cdot\nabla(\theta_\varepsilon\phi) + K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla(\theta_\varepsilon\phi)\right)dx + \\ \int_\Omega\left(K(\rho_\varepsilon)\rho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\theta_\varepsilon\phi + \operatorname{div}(\theta_\varepsilon\boldsymbol{v}_\varepsilon\phi)\int_0^{\rho_\varepsilon(x)} K(t)dt\right)dx = \\ \int_\Omega \mathbf{S}(\boldsymbol{v}_\varepsilon):\nabla\boldsymbol{v}_\varepsilon\phi dx.\end{aligned} \tag{5.39}$$

From (5.38) we see that

$$(1+\theta_\varepsilon^m)\frac{\varepsilon+\theta_\varepsilon}{\varepsilon}\nabla\theta_\varepsilon \rightharpoonup (1+\theta^m)\nabla\theta \qquad \text{in } L^1(\Omega)$$

and passing to the limit with the last four terms of LHS of (5.39) we get (using the strong convergence of the density)

$$\begin{aligned}\int_\Omega (-\rho\boldsymbol{v}\nabla(\theta\phi) - \rho\boldsymbol{v}\nabla(\theta\phi) + \rho\phi\boldsymbol{v}\nabla\theta + \operatorname{div}(\theta\phi\boldsymbol{v})\rho)\,dx \\ = \int_\Omega(-\rho\theta\boldsymbol{v}\cdot\nabla\phi + \rho\theta\operatorname{div}\boldsymbol{v}\phi)\,dx.\end{aligned} \tag{5.40}$$

To pass with the boundary term we have to use some interpolation argument (recall that $\theta_n \to \theta$ in any $L^q(\Omega)$):

$$\int_{\partial\Omega}|\theta|^{l+1} \le c\|\nabla\theta\|_q\|\theta\|^l_{\frac{lq}{q-1}},$$

which implies

$$\|\theta_n-\theta_m\|^{l+1}_{l+1,\partial\Omega} \le c\|\nabla(\theta_n-\theta_m)\|_q^{\frac{1}{l+1}}\|(\theta_n-\theta_m)\|^{l+\frac{l}{l+1}}_{\frac{lq}{q-1}},$$

where the first term is bounded and the second one converges – therefore the boundary term also converges.

The very last thing to do is to prove that the limit functions $\theta \in W^{1,p}(\Omega)$ and $\boldsymbol{v} \in W^{1,p}(\Omega)$. To make so we use the same method as in Subsection 4.2 – we use as test function

$$\Phi(\theta) = \int_0^\theta (1+t^m)dt$$

and we immediately see that $\theta \in L^\infty(\Omega)$ and $\boldsymbol{v} \in W^{1,p}(\Omega)$ for $p < \infty$. One more iteration in the energy equation and we see that $\theta \in W^{1,p}(\Omega)$ for $p < \infty$. This finally finishes the proof of Theorem 3.2.

6 Conclusions

In this whole work we focused our attention to steady Navier–Stokes–Fourier equations. The equations (including boundary slip conditions and constitutive equations) are stated in (1.1)–(1.13). They describe steady flow of compressible gas in a bounded two-dimensional domain $\Omega \in \mathcal{C}^2$.

We have proved that our problem has at least one weak solution, at least if we consider $\gamma > 2$ and $m > \frac{\gamma-1}{\gamma-2}$. The solution $(\rho, \boldsymbol{v}, \theta)$ is such that $\rho \in L^\infty(\Omega)$, $\boldsymbol{v} \in W^{1,q}(\Omega)$ and $\theta \in W^{1,q}(\Omega)$ for all $1 \le q < \infty$.

To get this fine result we used several methods from the mathematical theory of PDEs: we had to define an approximation to the original problem, using additional function $K(\rho)$ (defined in (4.1)), which was designed to allow us to prove some a priori estimates – in fact, we got density bounds immediately; on the other hand, later we had to show that under some conditions $K(\rho) \equiv 1$.

Next step was to prove existence results for the approximative problem. We did so with strong help of Leray–Schauder fixed-point theorem: We used the equations to construct two operators (defined in (4.45) and (4.47)). One of them helped us to prove the existence of the density, while the other was good to show that velocity field and temperature also exist. These two operators also gave us some idea about "regularity" of solution to the approximative problem and therefore we could proceed to prove that the solution of the approximative problem has everything in common with the solution of the original problem.

Next step was to prove the convergence. The convergence of the temperature was almost no problem, but the density was harder. We defined a quantity called effective viscous flux (see (5.19)) and showed that it is bounded and that it converges strongly in L^2. Then we used the effective viscous flux to prove strong convergence of the density (which was impossible earlier as we had only w*-convergence) and after that we concluded by showing the strong convergence of the velocity.

Therefore we found that the solution of the approximative system converges strongly to the solution of the original problem, which provided us sought information, and thus we proved Theorem 3.2.

Acknowledgement. The work has been supported by the grant of the Czech Science Foundation No. 201/08/0315.

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