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Computability of Branch-width of Submodular Partition Functions

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Foreword

The results presented in this work form one the chapters of my master thesis submitted at Charles University. I would like to thank Daniel Kráľ, my advisor, for introducing me to the concepts of submodular partition functions and tree-decompositions. Further, I thank him for fruitful discussions on the topic and careful readings of early versions of this work. The Sections 1 and 2 contain introduction to the subject and notation used throughout this work. In the Sections 3 and 4, my results on submodular partition functions are presented.

Abstract

The notion of submodular partition functions generalizes many of well-known tree decompositions of graphs. For fixed k, there are polynomial-time algorithms to determine whether a graph has tree-width, branch-width, etc. at most k. Contrary to these results, we show that there is no sub-exponential algorithm for determining whether the width of a given submodular partition function is at most two. In addition, we also develop another dual notion for submodular partition functions which is analogous to loose tangles for connectivity functions.

1 Introduction

Graph decompositions and width-parameters play a very important role in algorithmic graph theory (as well as structural graph theory). The most well-known and studied notions include the tree-width, branch-width and clique-width of graphs. The importance of these notions lie in the fact that many NP-complete problems can be decided for classes of graphs of bounded tree-/branch-width in polynomial time. A classical result of Courcelle [4] asserts that every problem expressible in the monadic second-order logic can be decided in linear time for the class of graphs with bounded tree-/branch-width. An analogous result for matroids with bounded branch-width representable over finite fields have been established by Hliněný [5, 6] and generalized using a more specialized notion of width to all matroids by Král' [8].

Most of the algorithms for classes of graphs of bounded width require a decomposition of an input graph as part of input. Fortunately, optimal tree-decompositions of graphs can be computed in linear time [2] if the width is fixed and there are even simple efficient approximation algorithms [3]. For branch-width, Oum and Seymour [9] recently established that the branch-decompositions of a fixed width of graphs and matroids can be computed in polynomial-time (or decided that they do not exist). Their algorithm actually deals with a more general notion of connectivity functions which are given by an oracle. A fixed-parameter algorithm for the same problem has been developed by Hliněný and Oum [7].

In this work, we study submodular partition functions introduced by Amini et al. [1]. This general notion includes both graph tree-width and branch-width as special cases. We postpone the formal definition to Section 2. In their paper, Amini et al. [1] presented a duality theorem that implies the known duality theorems for graph tree-width and graph/matroid branch-width of Robertson and Seymour [10].

Since the duality, an essential ingredient for some of the known algorithms for computing decompositions of small width, smoothly translates to this general setting, it is natural to ask whether decompositions of submodular partition functions with fixed width can be computed in polynomial-time. In this work, we show that such an algorithm cannot be designed in general. In particular, we present an argument that every algorithm deciding whether a partition width of an n-element set is at most two must ask an oracle the number of queries exponential in n. On a positive side, we were able to develop a notion of loose tangles, a key ingredient of the algorithm of Oum and Seymour [9], for this more general concept which we hope to be of some use to design algorithms for special classes of submodular partition functions.

2 Notation

In this section, we introduce the notation and concepts used in this work. A function $f : 2^E \to \mathbb{N}$ for a finite set E is said to be *submodular* if the following holds for every pair of subsets $X, Y \subseteq E$:

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) . \tag{1}$$

A submodular function f is *symmetric* if $f(X) = f(E \setminus X)$, for all subsets X of E. Finally, a *connectivity function* is a submodular function that is symmetric and $f(\emptyset) = 0$.

For a connectivity function f on a ground set E, a *branch-decomposition* of f is a pair (T, σ) where T is a *ternary* tree and σ is a bijection between the set of leaves of T and E. Every edge e of T naturally defines a bipartition $(A_e, \overline{A_e})$ of the ground set E, i.e., A_e consists of all elements that corresponds to leaves of T in one of the two components of $T \setminus e$. The *order* of an edge e of T is the value $f(A_e)$ and the *width* of a branch-decomposition (T, σ) is the maximum order of an edge of T. The *branch-width* of f is the minimum width of all branch-decompositions of f. This notion includes the notion of the usual branch-width of graphs and matroids.

There is a dual object to branch-decompositions called a *tangle*, introduced by Robertson and Seymour [10]. A set $\mathcal{T}$ of subsets of E is called an f-tangle of *order* $k + 1$ if $\mathcal{T}$ satisfies the following three axioms:

(T1) For all $A \subseteq E$, if $f(A) \leq k$, then either $A \in \mathcal{T}$ or $\overline{A} \in \mathcal{T}$.

(T2) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E$.

(T3) For all $e \in E$, we have $E \setminus \{e\} \notin \mathcal{T}$.

Robertson and Seymour [10] proved the following duality theorem between branch-decompositions and tangles.

Theorem 1 (Robertson and Seymour [10]). *Let f be a connectivity function on a ground set E. There is no f-tangle of order $k + 1$ if and only if the branch-width of f is at most k.*

We now introduce the concept of submodular partition functions that provides a unified view on branch-decompositions of connectivity functions and tree-decompositions of graphs. Throughout the work, Greek letters will be used for collections of subsets, i.e., α can stand for a collection $A_1, \ldots, A_k$ of subsets of a set E. Note, that the sets in a collection are not ordered in any way and a set can occur more than once in a collection. The collection α

is a *partition* if the sets A_i are mutually disjoint and their union is the whole set E.

There are shorthands for operations with collections of subsets we want to use: if α is such a collection $A_1, \ldots, A_k$ and A is another subset, then $\alpha \cap A$ stands for the collection $A_1 \cap A, \ldots, A_k \cap A$. We use $\alpha \setminus A$ in a similar way. Finally, $[A, \alpha]$ stands for the collection obtained from α by inserting A to the collection. Note that empty sets are allowed in the collections.

A *partition function* is a function from the set of all partitions to non-negative integers that satisfies $\psi([\emptyset, \alpha]) = \psi(\alpha)$ for every partition α, i.e., inserting an empty set to a collection does not change the value of the partition function. A partition function ψ is *submodular* if the following holds for every two partitions $[A, \alpha]$ and $[B, \beta]$:

$$\psi([A, \alpha]) + \psi([B, \beta]) \geq \psi([A \cup \overline{B}, \alpha \cap B]) + \psi([B \cup \overline{A}, \beta \cap A]) \qquad (2)$$

Similarly to branch-decompositions, Amini et al. [1] defined a *partitioning tree* of a partition function ψ. A *partitioning tree* on a finite set E is a tree T with a bijection σ between its leaves and E. Every internal node v of T corresponds to the partition of E whose parts are the leaves contained in subtrees of $T \setminus v$. A partitioning tree is *compatible* with a set of partitions $\mathcal{P}$ of E if all partitions corresponding to the internal nodes of T belong to $\mathcal{P}$.

Let $\mathcal{P}_k[\psi]$ denote the set of partitions α of E such that $\psi(\alpha) \leq k$. The *branch-width* of a submodular partition function ψ is the smallest integer k such that there exists a partitioning tree compatible with $\mathcal{P}_k[\psi]$. The concepts of submodular partition functions and partitioning trees include graph tree-width and branch-width as special cases.

There is a dual object to the partitioning tree called a *bramble* introduced by Amini et al. [1]. A *$\mathcal{P}$-bramble $\mathcal{B}$* on E is a set of pairwise intersecting subsets of E which contains a part of every partition of $\mathcal{P}$. A $\mathcal{P}$-bramble is called *non-principal* if it contains no singleton. The duality theorem for submodular partition functions asserts the following.

Theorem 2 (Amini et al. [1]). *Let ψ be a submodular partition function. There is no partitioning tree compatible with $\mathcal{P}_k[\psi]$ if and only if there is a non-principal $\mathcal{P}_k[\psi]$-bramble.*

3 Loose tangles

A key ingredient of the algorithm of Oum and Seymour [9] for deciding whether a connectivity function has branch-width k (k is fixed) is the notion of a *loose tangle* which we now recall. For a connectivity function f on a ground set E, a *loose f-tangle* of order $k+1$ is a set $\mathcal{T}$ of subsets of E satisfying the following three axioms:

(L1) $\emptyset \in \mathcal{T}$ and $\{e\} \in \mathcal{T}$ for every $e \in E$ such that $f(\{e\}) \leq k$.

(L2) If $A, B \in \mathcal{T}$, $C \subseteq A \cup B$, and $f(C) \leq k$, then $C \in \mathcal{T}$.

(L3) $E \notin \mathcal{T}$.

The following theorem by Oum and Seymour [9] states that the loose f-tangles are also dual objects to branch-decompositions of connectivity functions.

Theorem 3 (Oum and Seymour [9]). *Let f be a connectivity function on a ground set E. Then, no loose f-tangle of order $k+1$ exists if and only if the branch-width of f is at most k.*

Using loose tangles Oum and Seymour [9] managed to construct an algorithm for deciding whether the branch-width of a connectivity function is at most k for a fixed k in polynomial time when f is given by an oracle.

Similarly to the loose tangles of Oum and Seymour we introduce *loose tangles* for submodular partition functions. A *loose $\mathcal{P}$-tangle* is a set $\mathcal{T}$ of subsets of E closed under taking subsets satisfying the following three axioms.

(P1) $\emptyset \in \mathcal{T}$, $\{e\} \in \mathcal{T}$, for all $e \in E$ such that the partition $[\{e\}, \overline{\{e\}}]$ belongs to $\mathcal{P}$.

(P2) If $A_1, A_2, \ldots, A_p \in \mathcal{T}$, $C_i \subseteq A_i$, for $i = 1, \ldots, p$, $[C_1, \ldots, C_p, \overline{\cup_{i=1}^{p} C_i}] \in \mathcal{P}$, then $\cup_{i=1}^{p} C_i \in \mathcal{T}$.

(P3) $E \notin \mathcal{T}$.

To prove the main theorem of this section, we need a lemma.

Lemma 4. *Let ψ be a submodular partition function and $[A, \alpha]$ a partition. Then $\psi([A, \alpha]) \geq \psi([A, \overline{A}])$.*

Proof. Suppose that the partition $[A, \alpha]$ has at least three non-empty parts and let $[A, B, \beta] = [A, \alpha]$. By submodularity,

$$\begin{aligned}\psi([A, \alpha]) + \psi([B, \overline{B}]) &\geq \psi([A \cup \overline{B}, \alpha \cap B]) + \psi([B \cup \overline{A}, \overline{B} \cap A]) \\ &= \psi([B, \overline{B}]) + \psi([A, \overline{A}]).\end{aligned}$$

The result follows. □

In the following theorem, we show that for classes of partitions of bounded width, the loose tangle is a dual object to the partitioning tree.

Theorem 5. *Let ψ be a submodular partition function. There is no partitioning tree compatible with $\mathcal{P}_k[\psi]$ if and only if there is a loose $\mathcal{P}_k[\psi]$-tangle.*

Proof. Suppose there is a partitioning tree (T, σ) compatible with $\mathcal{P}_k[\psi]$ and a loose $\mathcal{P}_k[\psi]$-tangle $\mathcal{T}$. We will show that $\mathcal{T}$ violates (P3). Choose an arbitrary leaf x of T as a root. Every internal node v of T corresponds to a partition α_v. Let C_v be a union of all parts of α_v except the one containing x. Define C_v of a leaf v as the singleton $\sigma(v)$. We will show by backward induction on the distance from x that for every node v of T, the set C_v belongs to $\mathcal{T}$. Since T is a partitioning tree of E compatible with $\mathcal{P}_k[\psi]$, there is a partition $[\{e\}, \alpha_e]$ in $\mathcal{P}_k[\psi]$, for each $e \in E$. By Lemma 4, $\psi([\{e\}, \overline{\{e\}}]) \leq \psi([\{e\}, \alpha_e])$. Hence, $[\{e\}, \overline{\{e\}}]$ belongs to $\mathcal{P}_k[\psi]$ and $\{e\}$ is in $\mathcal{T}$ by (P1). For an inner node v, all his children $u_1, \ldots, u_p$ are farther from x than v and therefore all C_{u_i} are in $\mathcal{T}$. By (P2), since $[C_{u_i}, \overline{\cup C_{u_i}}]$ belongs to $\mathcal{P}_k[\psi]$, $C_v \equiv \cup C_{u_i} \in \mathcal{T}$. Finally, let v be the only child of x. Since $C_v \in T$ and $\{\sigma(x)\} \in \mathcal{T}$, by (P2), $C_v \cup \{\sigma(x)\} = E$ also belongs to $\mathcal{T}$. (P3) is now violated.

A partial partitioning tree for $A \subseteq E$ is a partitioning tree for a partition function ψ' on $(E \setminus A) \cup \{a\}$ defined as $\psi'([B, \beta]) = \psi(((B \setminus \{a\}) \cup A, \beta))$ for a partition $[B, \beta]$ where B contains a. We say that a set $A \subseteq E$ is k-branched if there is a partial partitioning tree for A compatible with $\mathcal{P}_k[\psi]$.

Define $\mathcal{T}$ to be a subset of 2^E closed under taking subsets, containing all singletons and all k-branched sets. We will show that $\mathcal{T}$ is a loose tangle. (P1) trivially holds since all k-branched singletons are in $\mathcal{T}$. Let $A_1, \ldots, A_p \in \mathcal{T}$ and $C_i \subseteq A_i$, $i = 1, \ldots, p$, such that $[C_1, \ldots, C_p, \overline{\cup C_i}] \in \mathcal{P}_k[\psi]$. We can assume that A_i are k-branched (otherwise take such a superset of it instead). Let $Y_1, \ldots, Y_p$, $Y_i \subseteq A_i$, be such sets that $\cup C_i \subseteq \cup Y_i$ and $\psi([Y_1, \ldots, Y_p, \overline{\cup Y_i}])$ is minimum. We will show that the set $\cup Y_i$ is k-branched.

To this end, we modify the partial partitioning tree T_i for A_i to be a partial partitioning tree for Y_i. At first, we delete from T_i all leaves corresponding to elements not in Y_i. We then repeatedly contract all nodes of degree two or less until we get a ternary tree T_i'. We claim T_i' is compatible with $P_k[\psi]$. Suppose for a contradiction that there is an internal node v' of T_i' corresponding to an internal node v of T_i such that $\alpha_{v'} \notin P_k[\psi]$. Assume $i = 1$ since we can relabel the parts so. Let $[A, \alpha] = \alpha_v$ such that A is the part of α_v that contains $\overline{A_1}$. We infer from the submodularity of the function ψ that

$$\begin{aligned}
&\psi([A, \alpha]) + \psi([Y_1, Y_2, \ldots, Y_p, \overline{\cup Y_i}]) \geq \psi([A \cup \overline{Y}_1, \alpha \cap Y_1]) \\
&+ \psi([Y_1 \cup \overline{A}, Y_2 \cap A, \ldots Y_p \cap A, \overline{\cup Y_i} \cap A])
\end{aligned}$$

The choice of $Y_1, \dots, Y_p$ yields that

$$\psi([Y_1 \cup \overline{A}, Y_2 \cap A, \dots, Y_p \cap A, \overline{\cup Y_i} \cap A]) \geq \psi([Y_1, \dots, Y_p, \overline{\cup Y_i}]).$$

Hence, $\psi([A \cup \overline{Y}_1, \alpha \cap Y_1]) \leq \psi([A, \alpha]) \leq k$ and T_1' is compatible with $\mathcal{P}_k[\psi]$.

Now, construct a partial partitioning tree T by connecting T_i' to a single node corresponding to a partition $[Y_1, \dots, Y_p, \overline{\cup Y_i}]$. This partition belongs to $\mathcal{P}_k[\psi]$ since $\psi([Y_1, \dots, Y_p, \overline{\cup Y_i}]) \leq \psi([C_1, \dots, C_p]) \leq k$ by the minimality of $\psi([Y_1, \dots, Y_p, \overline{\cup Y_i}])$. Therefore T is a partial partitioning tree for $\cup Y_i$ compatible with $\mathcal{P}_k[\psi]$ and thus $\cup Y_i \in \mathcal{T}$. Since $\cup C_i \subseteq \cup Y_i$, also $\cup C_i \in \mathcal{T}$ as required.

If $E \in T$, then E is k-branched and the partial partitioning tree for E is actually a partitioning tree for ψ. This contradicts the fact that ψ does not have a partitioning tree compatible with $\mathcal{P}_k[\psi]$. Therefore, $E \notin \mathcal{T}$ and (P3) holds. We conclude that $\mathcal{T}$ is a loose $\mathcal{P}_k[\psi]$-tangle. □

4 Hardness of submodular partition functions

We first have to define several auxiliary functions before we can establish our hardness result. Let g_n be the function $g_n : 2^E \to \mathbb{N}$ for $E = \{1, \dots, 2n\}$ defined as $g_n(X) = \min\{|X|, |\overline{X}|\}$. We start our exposition with showing that g_n is submodular.

Lemma 6. *The function g_n is submodular for every n.*

Proof. Consider two subsets X and Y. If both $|X| \leq n$ and $|Y| \leq n$, then

$$\begin{aligned} g_n(X) + g_n(Y) &= |X| + |Y| = |X \cap Y| + |X \cup Y| \\ &\geq g_n(X \cap Y) + g_n(X \cup Y). \end{aligned}$$

If both $|X| > n$ and $|Y| > n$, we get the same result by the symmetry of g.

$$\begin{aligned} g_n(X) + g_n(Y) &= g_n(\overline{X}) + g_n(\overline{Y}) \geq g_n(\overline{X} \cap \overline{Y}) + g_n(\overline{X} \cup \overline{Y}) \\ &= g_n(X \cup Y) + g_n(X \cap Y) \end{aligned}$$

So suppose that $|X| > n$ and $|Y| \leq n$. We get

$$\begin{aligned} g_n(X) + g_n(Y) &= |\overline{X}| + |Y| = |\overline{X} \setminus Y| + |Y \setminus \overline{X}| + 2|\overline{X} \cap Y| \\ &\geq g_n(\overline{X} \setminus Y) + g_n(Y \setminus \overline{X}) = g_n(\overline{X} \cap \overline{Y}) + g_n(X \cap Y) \\ &= g_n(X \cup Y) + g_n(X \cap Y). \end{aligned}$$

This finishes the proof. □

The function g_n can be extended to a partition function ϕ_n on the ground set $E = \{1, \ldots, 2n\}$ by setting

$$\phi_n(\alpha) = \max_{i \in I} g_n(A_i).$$

A part A_i of α is *dominating* if $g_n(A_i) = \phi_n(\alpha)$. Note that, if α has a part with at least n elements, then that part is dominating.

We proceed by showing that the function ϕ_n is submodular.

Lemma 7. *The function ϕ_n is submodular for every n.*

Proof. We check the following inequality for all partitions $[A, \alpha]$ and $[B, \beta]$:

$$\phi_n([A, \alpha]) + \phi_n([B, \beta]) \geq \phi_n([A \cup \overline{B}, \alpha \cap B]) + \phi_n([B \cup \overline{A}, \beta \cap A]).$$

Observe that at least one of the parts $A \cup \overline{B}$ or $B \cup \overline{A}$ in this inequality is dominating since one of $A, \overline{A}$ and one of $B, \overline{B}$ has at least n elements. If both $A \cup \overline{B}$ and $B \cup \overline{A}$ are dominating, then the submodularity of ϕ_n follows from the submodularity of g:

$$\begin{aligned}
\phi_n([A, \alpha]) + \phi_n([B, \beta]) &\geq g_n(A) + g_n(B) = g_n(A) + g_n(\overline{B}) \\
&\geq g_n(A \cap \overline{B}) + g_n(A \cup \overline{B}) = g_n(\overline{A} \cup B) + g_n(A \cup \overline{B}) \\
&= \phi_n([A \cup \overline{B}, \alpha \cap B]) + \phi_n([B \cup \overline{A}, \beta \cap A])
\end{aligned}$$

Suppose that $A \cup \overline{B}$ is not dominating, so take an $A_i \in \alpha$ such that $A_i \cap B$ is dominating. Since $|B| \geq n$ and $A_i \subseteq \overline{A}$, it holds that $g_n(A_i \cup B) \geq g_n(B \cup \overline{A})$. We use this inequality to prove the submodularity as follows:

$$\begin{aligned}
\phi_n([A, \alpha]) + \phi_n([B, \beta]) &\geq g_n(A_i) + g_n(B) \geq g_n(A_i \cap B) + g_n(A_i \cup B) \\
&\geq g_n(A_i \cap B) + g_n(B \cup \overline{A}) \\
&= \phi_n([A \cup \overline{B}, \alpha \cap B]) + \phi_n([B \cup \overline{A}, \beta \cap A])
\end{aligned}$$

The case when $B \cup \overline{A}$ is not dominating follows by symmetry. $\square$

Values of the function ϕ_n range between 0 and n. We now truncate the function and define the following partition function $\phi_{n,k}$ on $E = \{1, \ldots, 2n\}$ as follows:

$$\phi_{n,k}(\alpha) = \min\{\phi_n(\alpha), k\}.$$

Next, we show that the function ϕ_n stays submodular after the truncation.

Lemma 8. *The function $\phi_{n,k}$ is submodular for every n and k.*

Proof. Let us consider two partitions $[A,\alpha]$ and $[B,\beta]$ that violates the inequality (2):

$$\phi_{n,k}([A,\alpha]) + \phi_{n,k}([B,\beta]) \geq \phi_{n,k}([A\cup\overline{B}, \alpha\cap B]) + \phi_{n,k}([B\cup\overline{A}, \beta\cap A]).$$

Since $\phi_{n,k}(\gamma) \leq \phi_n(\gamma)$ for all partitions γ, at least one of $\phi_n([A,\alpha])$ or $\phi_n([B,\beta])$ is larger than k. If both of them are, then the inequality trivially holds. Suppose that $\phi_n([A,\alpha]) < k$. We will show that at least one of $\phi_n([A\cup\overline{B}, \alpha\cap B])$ or $\phi_n([B\cup\overline{A}, \beta\cap A])$ is smaller or equal to $\phi_n([A,\alpha])$.

If $|A| \geq n$, then $\phi_n([A\cup\overline{B}, \alpha\cap B]) \leq \phi_n([A,\alpha])$ since $A\cup\overline{B}$ is the dominating part and $g_n(A\cup\overline{B}) \leq g_n(A) \leq \phi_n([A,\alpha])$. If $|A| < n$, then $\phi_n([B\cup\overline{A}, \beta\cap A]) \leq \phi_n([A,\alpha])$ since $B\cup\overline{A}$ is the dominating part and $g_n(B\cup\overline{A}) \leq g_n(\overline{A}) \leq \phi_n([A,\alpha])$. This finishes the proof. □

Now, we use the function $\phi_{n,3}$ to construct partition functions ϕ^*_n and $\phi^*_{n,\beta}$ which appear in our hardness result. The function ϕ^*_n is defined as

$$\phi^*_n(\alpha) = \begin{cases} \phi_{n,3}(\alpha) & \text{if } \alpha \text{ has at most three non-empty parts, and} \\ 3 & \text{otherwise.} \end{cases}$$

For a partition β of $\{1,\ldots,2n\}$ into n two-element subsets, the function $\phi^*_{n,\beta}$ is then defined as

$$\phi^*_{n,\beta}(\alpha) = \begin{cases} \phi_{n,3}(\alpha) & \text{if } \alpha \text{ has at most three non-empty parts,} \\ 2 & \text{if } \alpha = \beta\text{, and} \\ 3 & \text{otherwise.} \end{cases}$$

First, we show that these functions are submodular.

Lemma 9. *The function ϕ^*_n is submodular for every n.*

Proof. Observe the following:

- If $\phi(\alpha) = 0$, then also $\phi^*_n(\alpha) = 0$.
- If $\phi(\alpha) = 1$, then $\phi^*_n(\alpha) = 1$ unless α is a set of singletons where $\phi^*_n(\alpha) = 3$.
- If $\phi(\alpha) = 2$, then $\phi^*_n(\alpha) = 2$ unless α has more than three non-empty parts. In this case, every part of α is a pair or a singleton.

Therefore the functions $\phi_{n,3}$ and ϕ^*_n differ only on partitions consisting of singletons and pairs.

Let us assume for a contradiction that ϕ^*_n is not submodular. Since $\phi^*_n(\alpha) \geq \phi_{n,3}(\alpha)$ for all partitions α, the violation of the submodularity is caused by an increase on the right-hand side of (2). Consider partitions $[A, \alpha]$ and $[B, \beta]$ violating the inequality (2). Hence, say, $\gamma = [A \cup \overline{B}, \alpha \cap B]$ is that partition containing only singletons and pairs. Since γ has all parts of size at most two, $|\overline{B}| \leq 2$. If $\overline{A} \cap \overline{B} = \emptyset$, then $\overline{B} \subseteq A$ and $\overline{A} \subseteq B$. Therefore $\gamma = [A, \alpha]$, $[B \cup \overline{A}, \beta \cap A] = [B, \beta]$ and the inequality trivially holds. So we can assume that $|B \cup \overline{A}| > |B|$ and since $2n-2 \leq |B| < 2n$, by the definition of ϕ^*_n

$$\phi^*_n([B, \beta]) > \phi^*_n([B \cup \overline{A}, \beta \cap A]) . \tag{3}$$

Since the number of non-empty parts of γ is at least 4, the number of non-empty parts of $[A, \alpha]$ is at least 3 and therefore $\phi^*_n([A, \alpha]) \geq 2$ by the definition of ϕ^*_n. The submodularity follows from (3) and the fact that $\phi^*_n(\gamma) \leq 3 \leq \phi^*_n([A, \alpha]) + 1$. □

Lemma 10. *The function $\phi^*_{n,\beta}$ is submodular for every $n \geq 4$ and for every partition β consisting only of two-element sets.*

Proof. Since ϕ^*_n and $\phi^*_{n,\beta}$ differ only on partition β where $\phi^*_n(\beta) \geq \phi^*_{n,\beta}(\beta)$, β has to be on the left-hand side of the inequality (2) to violate it. Let $[A, \alpha]$ and $\beta = [C, \gamma]$ be the partitions violating the inequality (2):

$$\phi^*_{n,\beta}([A, \alpha]) + \phi^*_{n,\beta}([C, \gamma]) \geq \phi^*_{n,\beta}([A \cup \overline{C}, A \cap C]) + \phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A])$$

Since $|C| = 2$, $\phi^*_{n,\beta}([A \cup \overline{C}, A \cap C]) \leq 2$. Hence $\phi^*_{n,\beta}([A, \alpha]) \leq 2$. If $|A| \leq 2$, then $|C \cup \overline{A}| \geq 2n - |A|$ and $\phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A]) \leq \phi^*_{n,\beta}([A, \alpha])$, contradicting the assumption. Therefore A has to have at least $2n - 2$ elements and $\phi^*_{n,\beta}([A \cup \overline{C}, A \cap C]) \leq \phi^*_{n,\beta}([A, \alpha])$.

If $\overline{C} \subseteq A$, then $\overline{A} \subseteq C$ and $\phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A]) = \phi^*_{n,\beta}([A, \alpha])$, contradicting the assumption. Therefore $|A \cup \overline{C}| > |A|$ giving $\phi^*_{n,\beta}([A, \alpha]) > \phi^*_{n,\beta}([A \cup \overline{C}, A \cap C])$. Since $\phi^*_{n,\beta}(\beta) + 1 = 3 \geq \phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A])$, the inequality (2) holds — a contradiction. □

In the proof of the main theorem we will use the fact that the width of the function ϕ^*_n is three while the width of the modified function $\phi^*_{n,\beta}$ is two. To see that branch-width of $\phi^*_{n,\beta}$ is at most two, just consider the following branch-decomposition T of $\phi^*_{n,\beta}$. T has a root x with n children $v_1, \ldots, v_n$ each v_i connected to two leaves corresponding to the two elements in β_i. Since $\phi^*_{n,\beta}(\alpha_x) = \phi^*_{n,\beta}(\beta) = 2$ and $\phi^*_{n,\beta}(\alpha_{v_i}) = 2$, for $i = 1, \ldots, n$, the branch-decomposition T has width two. In the next lemma, we show that the branch-width of ϕ^*_n is three.

Lemma 11. *For $n \geq 4$, the branch-width of ϕ^*_n is three.*

Proof. Let T be a branch-decomposition of ϕ_n^* of width smaller than three. We assume there are no nodes of degree two in T since we can contract them obtaining a smaller branch-decomposition of the same width. Since every internal node v of T of degree larger than three corresponds to a partition α_v of E with more than three parts (thus $\phi_n^*(\alpha_v) = 3$), there are no such vertices in T and T is a ternary tree. Consider an arbitrary internal node v of T with less than two leaves as neighbors. There have to be such a vertex v since there are at most n vertices with two leaves as neighbors but there are $2(n-1)$ internal nodes. For such a vertex v, α_v contains a part with at least three elements and at most $2n-3$ elements implying $\phi_n^*(\alpha_v) = 3$. This finishes the proof. $\square$

We are now ready to establish our hardness result. We assume the existence of an algorithm and show that it cannot discover a small discrepancy between a submodular partition function having width three and two.

Theorem 12. *There is no sub-exponential algorithm for determining whether the branch-width of an oracle-given submodular partition function on a set with $2n$ elements is at most two.*

Proof. Assume that there exists such a sub-exponential algorithm $\mathcal{A}$ and run $\mathcal{A}$ for the submodular partition function ϕ_n^*. The algorithm $\mathcal{A}$ must clearly output that the width ϕ_n^* is at least three. Since the running time of the algorithm is sub-exponential, for n sufficiently large, there exists a partition β of $\{1, \ldots, 2n\}$ into n two-element subsets such that $\mathcal{A}$ never queries β (there are $(2n)!/(n!2^n)$ such partitions and $\mathcal{A}$ cannot query all of them because of its running time). However, the algorithm $\mathcal{A}$ for $\phi_{n,\beta}^*$ performs the same steps and thus it outputs that the width of $\phi_{n,\beta}^*$ is at least three which is not correct. $\square$

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