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Modelování nestlačitelných ionizovaných směsí

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Abstract: In the present work we study the model of incompressible ionized mixtures from point of view of mathematical analysis and practical implementations. The model consists of the continuity equation, Navier-Stokes equations, Nernst-Planck equations and Poisson equation for the electric potential. We proove the existence of a weak solution for the model in three space dimensions.

Keywords: partial differential equations, fluid mixtures, Navier-Stokes equation, Nernst-Planck equation, Poisson equation

1 Introduction

The mixture theory has many interesting applications. Let us give an incomplete list of them:

- modelling biological systems,
- designing and interpreting electrophoresis experiments,
- designing chemical reactors,
- optimizing fuel cells

At least some of these processes are assumed to be fitted to incompressible model. Contrary to works [Rou06], [Rou05], [Rou07] and [Gla07] we would rather assume the incompressibility only for the whole mixture. The main problem is that incompressibility should not be confused with volume additivity hypothesis. Let us for example consider water and sugar. We will agree that both are incompressible substances. Still while mixing the mixture gets "compressed". However, once mixed, the mixture may be considered as incompressible again. Very important role in modelling of the mixture behavior plays the electric potential. In biological cells may the electric field attain intensity 10-100 MV/m. During the electrophoresis may the electric field cause almost full separation of components.

The model proposed here is based on so called Eckart-Prigogine principle (see [Pri47] and [Eck40]) introducing barycentric velocity. This principle allows us to avoid any in principle unmeasurable quantities which rational thermodynamics cannot avoid. New approach of beyond equilibrium thermodynamics is also refered. Mathematical treatment of models with density non-constant in space is more difficulty and techniques used in [Gla07] to get convergence of semi-implicit method. The most important difficulty is obtaining maximum principle. The maximum principles cannot be obtained on discrete systems but also cannot be obtained aposteriori on fully coupled equations due to insufficient regularity of density.

2 Deduction of the model

In this section we try to deduce a model of incompressible ionized mixtures which is mathematically treatable but still captures several physical phenomena. Main improvement in comparison with [Rou06] is removing volume-additivity hypothesis and the assumption that all species in the mixtures have the same specific densities. For the sake of simplicity we assume that

the density of i-th species	ρ_i
the velocity of i-th species	v_i
the reaction rates vector	r
the density	ρ
the barycentric velocity	v
the vector of mass fractions	c
the diffusion flux	j_i
the free energy	ψ
the electrochemical potential	μ_i
the matrix of transport coefficients	M,$\mathbf{\Lambda}^c$
the diffusion matrix	D
the electro-mobility vector	m
the diffusion coefficient	D_i
the specific charge	z_i
the trajectory	χ
the stress tensor	$\mathbb{T}$
the pressure	p
the electric potential	ϕ
the stress tensor	$\mathbb{T}$
the viscosity	ν
the charge density	ρq
the permittivity	ϵ
the activity	a_i
the time step	τ
the regularization parameter	ε
the test function	φ
the domain	Ω
the time interval	I
the time-space cylinder	$Q = I \times \Omega$
the boundary of the domain times the time interval	$\Sigma = I \times \partial\Omega$

Table 1: the nomenclature

all functions in the model are C^∞ . This assumption is quite usual even when deriving balance equations for single component fluid but may be relaxed. For relaxation of the smoothness assumption in single component case see [Fei03]. The procedure used here suffers several problems that are also listed. The last but not least goal of this section is to relate the model to other models used in the literature.

Let us derive the model. First we formulate the continuity equation for every species:

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i v_i) = r_i \tag{2.1}$$

Here we denoted by v_i the velocity of the species, ρ_i the density of the species in the mixture and r_i the reaction rate. Let us note that we have that $\sum_{i=1}^{L} r_i = 0$ by conservation of mass in reactions. In rational thermodynamics we would proceed by formulating other balance equations for each species (for this approach see for example [TT60]). Here we proceed by using the concept of barycentric velocity going back to the work of Eckart and Prigogine. We introduce the following notation:

- $\rho = \sum_{i=1}^{L} \rho_i$ is the density of the whole mixture,
- $c_i = \frac{\rho_i}{\rho}$ is the mass fraction of i-th species,
- $v = \sum_{i=1}^{L} c_i v_i$ is the barycentric velocity,
- $j_i = \rho_i v_i - \rho v$ is the diffusion flux.

From definition we see that $\sum_{i=1}^{L} j_i = 0$. By summing the equations (2.1) we get the the usual continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0\,. \tag{2.2}$$

We shall also rewrite the equation (2.1) to get rid of v_i

$$\frac{\partial \rho c_i}{\partial t} + \operatorname{div}(\rho c_i v - j_i) = r_i\,. \tag{2.3}$$

Our main goal is now to find a constitutive equation for j_i. We choose it here as

$$j_i = \sum_{j=1}^{L} M_{ij} \nabla \mu_j\,.$$

The quantity μ_j is the electrochemical potential and is defined as the derivative of free energy ψ with respect to c_i. However, from mathematical point of view the electrochemical potentials are not good functions to work with because it is hard to get some estimates on them. We shall now assume that $\psi = \psi(c, \phi)$ and use the chain rule. We obtain

$$j_i = \sum_{j=1}^{L} M_{ij} \frac{\partial^2 \psi}{\partial c_j^2} \nabla c_j + \sum_{j=1}^{L} M_{ij} \frac{\partial^2 \psi}{\partial c_i \partial \phi} \nabla \phi \,. \tag{2.4}$$

To simplify the notation we define

$$D_{ij} = M_{ij} \frac{\partial^2 \psi}{\partial c_j^2} \tag{2.5}$$

and

$$m_i = \sum_{j=1}^{L} M_{ij} \frac{\partial^2 \psi}{\partial c_i \partial \phi} \tag{2.6}$$

The matrix M can be chosen according to [dG80] or [GNS04] as

$$M_{ij} = D_i c_i \left(\delta_{ij} - \frac{D_j c_j}{\sum_{k=1}^{L} D_k c_k} \right)$$

where D_i is the diffusion coefficient according to i-th species. Denoting z_i specific charge of i-th species and $q = \sum_{i=1}^{L} z_i c_i$ we may write a constitutive relation

$$\psi(c) = \sum_{i=1}^{L} K c_i \log c_i + q\phi \,. \tag{2.7}$$

Using this relation we deduce

$$D_{ij} = K D_i \left(\delta_{ij} - \frac{D_j c_j}{\sum_{k=1}^{L} D_k c_k} \right)$$

and

$$m_i = K D_i c_i \left(z_i - \frac{\sum_{k=1}^{L} D_k z_k c_k}{\sum_{k=1}^{L} D_k c_k} \right) .$$

The incompressibility and the momentum balance we formulate for the mixture as whole. For deriving of incompressibility condition we define $\chi(t,x)$ by equation

$$\frac{\partial \chi(t,x)}{\partial t} = v(t, \chi(t,x)) \,, \quad \chi(0,x) = x \,. \tag{2.8}$$

In one component fluid moving by velocity v this has the meaning of change of variables into Lagrangian coordinates. Incompressibility condition says that volume is preserved by flow, ie.

$$\int_K \mathrm{d}x = \int_{\chi(t,K)} \mathrm{d}x = \int_K \det \nabla\chi(t,x)\mathrm{d}x$$

The condition is equivalent to the condition $\det \nabla\chi(t,x) = 1$. Taking the time derivative

$$0 = \frac{\partial \det \nabla\chi(t,x)}{\partial t} = \operatorname{div} v \det \nabla\chi(t,x)$$

we conclude that $\operatorname{div} v = 0$. The momentum balance is expressed by well known Navier-Stokes equation driven by the Lorentz force on the right hand side.

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v - \mathbb{T}) + \nabla p = \rho q \nabla\phi$$

The last relation we need is the equation for the electric potential. Full Maxwell equations here reduce to a Poisson equation

$$-\epsilon\Delta\phi = \rho q\,.$$

The resulting system is

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v - \mathbb{T}) + \nabla p = \rho q \nabla\phi \ , \ \operatorname{div} v = 0\,, \tag{2.9a}$$

$$\partial_t \rho + \operatorname{div}(\rho v) = 0\,, \tag{2.9b}$$

$$\partial_t \rho c_i + \operatorname{div}\left(\rho v c_i - \sum_{j=1}^{L} D_{ij}(c)\nabla c_j - m_i(c)\nabla\phi\right) = r_i(c)\,, \tag{2.9c}$$

$$-\operatorname{div}(\epsilon\nabla\phi) = \rho q\,, \tag{2.9d}$$

where $q = c \cdot z$ is the charge, z are specific charges of constituents and

$$\mathbb{T} = 2\nu(c)\mathrm{D}v \tag{2.9e}$$

is the stress tensor. The boundary conditions are:

$$v = 0\,, \tag{2.9f}$$

$$(D(c)\nabla c + m(c)\nabla\phi)\mathbf{n} = 0\,, \tag{2.9g}$$

$$\frac{\partial\phi}{\partial n} = (\phi - \phi_\Sigma) \tag{2.9h}$$

and the initial conditions are

$$\rho(0) = \rho_0 \,, \tag{2.9i}$$
$$v(0) = v_0 \,, \tag{2.9j}$$
$$c(0) = c_0 \,. \tag{2.9k}$$

Once derived the model let us discuss the relation of the models and other model used in chemistry. Chemists tend to use another type of variables – analytic concentrations defined as number of moles divided by volume or activities. However, models using analytical concentrations are equivalent to the model described here by the formula $\rho_i = M_i \tilde{c}_i$ where $\tilde{c}_i$ are analytical concentrations and M_i are molar weights. The activities are quantities defined by formula

$$\psi(a, \phi) = \sum_{i=1}^{L} K a_i \log a_i + q\phi \,. \tag{2.10}$$

The relation between any type of concentrations and activities is a bit obscure, in fact can be precisely written only in terms of implicit functions as $\psi(c, \phi) = \psi(a(c), \phi)$. For needs of numerical computations there exists some approximate formulae defining $c = c(a)$. The reason for introducing activities was discrepancy between constitutive relation (2.7) and real forms of free energy. As we saw in (2.4) the form of the model is not affected by choice of free energy and there is no need to introduce another experimentally hardly measurable or even unmeasurable quantity called activity.

In chemistry one also often uses modelling assumptions of local chemical equilibrium and electroneutrality. In mathematical language it means that the mixture evolves in such way that

$$r_i(c) = 0 \text{ and } q(c) = 0 \,. \tag{2.11}$$

The equations should be interpreted as nonlinear constraints replacing some equations of (2.3). It is well known even among chemists that such equations admit non-unique solution, but this fact is quietly ignored as only one solution is chemically admissible (for example in terms of pH). To the author's knowledge the notion of chemical admissibility in not axiomatically defined and depends only on chemist's intuition. Our model could be theoretically constructed to preserve chemical equilibria and electroneutrality but the analysis done in section 3 based on positive definiteness of matrix D would fail. Chemical equilibrium and electroneutrality is in the general case only description of some stationary state of the equation. We also may give some precise meaning to the notion of chemical admissibility in terms of stability of the stationary solution.

Let us in short mention other models appeared in literature. Very popular is the rational thermodynamics for mixtures introduced by [TT60] and already mentioned here. In spite of their popularity these results in mixture theory are not as nice as one would await. The best results to the authors knowledge are [Sam07]. New and promising approach is beyond equilibrium thermodynamics.Also interesting but mathematically untreatable is model proposed in [EPM97]. This model replaces the condition $\sum_{i=1}^{L} j_i = 0$ by the the condition

$$\operatorname{div} \sum_{i=1}^{L} j_i = 0 \tag{2.12}$$

and defines

$$j_i = D_i c_i \nabla(\mu_i + p)$$

with p chosen such that (2.12) holds. It is shown [OE97] that the model is in some situations more accurate our model with matrix computed in [dG80]. Our model can in compressible case arise also from Boltzmann equation as shown in [Gio99]. Let us finish this excursion by noting that many more or less rigorously derived models arise in studies of fuel cells. A good summary of the models is in [WN04].

The deduchion suffers several problems. Let us look at some of them. The first one is that the model assumes quietly that no separation of species takes place. Formally this assumption is inserted in the assumption that v_i is defined on whole Ω. The more practical side of the problem is that if separation of species takes place, the surface tension has to be considered. One of possible solutions to such problem may be to introduce the free energy in the form of $\psi = \psi(\rho, c, \nabla c, \phi)$ as was done in [AF07].

The assumption that the free energy is independent of the density also is not too realistic. To illustrate this we recall formula for the pressure compressible case $p = \rho^2 \frac{\partial \psi}{\partial \rho}$. The main reason for omitting the dependency is that$\nabla \rho$ which would arise by the chain rule in (2.4) would be untreatable as there is really no reason for density in weak formulation to be differentiable.

The last objection is against the meaning of χ from (2.8). In single component fluid it has $\chi(\cdot, x)$ the meaning of trajectory of a material point which was initially in position x. This notion is at least controversial in mixtures and we may wonder whether the characterisation of incompressibility by term $\operatorname{div} v = 0$ is really the thing we need. The only argument we have to advocate characterising incompressibility by the condition $\operatorname{div} v = 0$ is the fact that if $v \in C$ and $\operatorname{div} v = 0$ we may solve the continuity equation by the function $\rho(t, \chi(t, x)) = \rho(0, x)$ and observe that the fluid really do no get compressed.

3 Existence proof

We will study the equations (2.9). The data qualification is:

- $\nu \in C^1$, $0 < \nu_l \leq \nu(c) \leq \nu_u$,
- $D \in C^\infty$, D positive definite on $\{x \in \mathbb{R}^L : \sum_{i=1}^L x_i = 0\}$, $\sum_{i=1}^L D_{ij} = 0$ and for $c_i \leq 0$ we have $D_{ii}(c) > 0$ and $D_{ij}(c) = 0$ for $i \neq j$,
- $m \in C^\infty$ and for $c_i \leq 0$ holds $m_i(c) = 0$.
- $r \in C$ bounded and for $c_i \leq 0$ we have $r_i(c) \geq 0$,
- $\epsilon > 0$,
- $\phi_\Sigma \in L^2(\partial\Omega)$,

We will follow techniques developed by Abels and Feireisl in [AF07]. Contrary to the cited article this theses studies the incompressible case and coupling with the electric potential but does not include any gradients in the free energy. The Nernst-Planck equations are formulated and treated as in [Rou06].

Now let us define, what a solution means.

Definition 3.1. As a weak solution to the system (2.9) we will call

- $\rho \in L^\infty(I \times \Omega; \mathbb{R})$,
- $v \in L^2(I; W^{1,2}_{\mathrm{div}}(\Omega; \mathbb{R}^3))$,
- $c \in L^2(I; W^{1,2}(\Omega; \mathbb{R}^L))\,,\ \sum_{i=1}^L c_i = 1\,,\ c_i \geq 0$,
- $\phi \in L^\infty(I; W^{1,2}(\Omega; \mathbb{R}^L)$

satisfying

1. with every $\varphi \in W^{1,2}(Q)$, $\operatorname{div}\varphi = 0$, $\varphi(T) = 0$

$$\begin{aligned}&\int_Q \rho v \partial_t \varphi + (\rho v \otimes v + \mathbb{T}) : \nabla\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega \rho_0 v_0 \varphi(0)\mathrm{d}x\\&= \int_Q \rho q \nabla\phi\varphi \mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.1}$$

2. ρ is a renormalized solution of continuity equation. This means that for every $\varphi \in C_0^\infty((0,T) \times \bar{\Omega})$ and every $b \in C^0([0;\infty))$ bounded with

$$B(\rho) = B(1) + \int_1^\rho \frac{b(z)}{z^2} \mathrm{d}z$$

satisfy

$$\int_Q (\rho B(\rho) \partial_t \varphi + \rho B(\rho) v \cdot \nabla \varphi - b(\rho) \operatorname{div} v \varphi) \mathrm{d}x \mathrm{d}t = 0 \tag{3.2}$$

3. the concentrations satisfy with every $\varphi \in W^{1,2}(Q)$, $\varphi(T) = 0$

$$\begin{aligned} \int_Q \rho c_i \partial_t \varphi + (\rho c_i v + \sum_{j=1}^{L} D_{ij}(c) \nabla c_j + m_i(c) \nabla \phi) \cdot \nabla \varphi \mathrm{d}x \mathrm{d}t \\ \int_\Omega \rho(0) c(0) \varphi(0) \mathrm{d}x = - \int_Q r_i(c) \varphi \mathrm{d}x \mathrm{d}t \end{aligned} \tag{3.3}$$

4. with every $\varphi \in L^1(I; W^{1,2}(\Omega))$

$$\int_Q \epsilon \nabla \phi \cdot \nabla \varphi \mathrm{d}x \mathrm{d}t + \int_\Sigma \epsilon (\phi - \phi_\Sigma) \varphi \mathrm{d}x \mathrm{d}t = \int_Q \rho q \varphi \mathrm{d}x \mathrm{d}t \tag{3.4}$$

Remarque 3.2. The concept of a renormalized solution of the continuity equation was introduced by DiPerna and Lions in their article [DL89].

For the proof of existence of such solution we will use four approximation steps:

- regularization of the continuity equation by term $\varepsilon \Delta \rho$,
- time discretization by Roethe method,
- space discretization of Navier-Stokes and Poisson equations by Galerkin method,
- decoupling into linear equations.

The proof is organized as follows:

1. The existence for regularized, discretized and decoupled equations is proved.
2. We prove the maximum principle for continuity equation and the estimates uniform in time.

3. We use the Shauder's fixed-point theorem to prove existence of solution for regularized and discretized system.

4. The rest of apriori estimates are done.

5. The limit passage from discretization gives us the solution of regularized system.

6. We pass to the limit from the approximation.

For the discretization in space we introduce two sequences of finite dimensional spaces

$$V^n_{0,\mathrm{div}} \subset C^1_{0,\mathrm{div}}(\Omega) \ , \ V^n \subset C^1(\Omega)$$

which satisfy $\overline{\bigcup_n V^n_{0,\mathrm{div}}}^{\|\cdot\|_{W^{1,2}(\Omega)}} = W^{1,2}_{0,\mathrm{div}}(\Omega)$ and $\overline{\bigcup_n V^n}^{\|\cdot\|_{W^{1,2}(\Omega)}} = W^{1,2}(\Omega)$. For time discretization we introduce the sequence of timesteps $\tau^n \to 0$. We assume that $\frac{T}{\tau^n} = N(\tau^n) \in \mathbb{N}$. For fixed sequences of spaces $V^n_{0,\mathrm{div}}$ and V^n the terms of the sequence τ_n should be chosen small enough. The exact condition will arise during the proof and in case of finite element discretization will have the meaning of the stability condition. We also define retract

$$K(c)_i = \frac{c_i^+}{\sum_{j=1}^L c_j^+}$$

First we approximate the continuity equation to be able to test by solutions themselves. The approximated continuity equation is

$$\partial_t \rho_\varepsilon + \mathrm{div}(\rho_\varepsilon v_\varepsilon) = \varepsilon \Delta \rho_\varepsilon \, .$$

The approximation also affects Navier-Stokes and Nernst-Planck equations

$$\begin{aligned}
&\partial_t(\rho_\varepsilon v_\varepsilon) + \varepsilon \nabla \rho_\varepsilon \nabla v_\varepsilon + \mathrm{div}\left(\rho v_\varepsilon \otimes v_\varepsilon - \mathbb{T}(c_\varepsilon, \mathrm{D} v_\varepsilon)\right) + \nabla p_\varepsilon = \rho q(c_\varepsilon) \nabla \phi_\varepsilon \, , \\
&\partial_t(\rho_\varepsilon (c_\varepsilon)_i) - \varepsilon (c_\varepsilon)_i \Delta \rho_\varepsilon \\
&\quad + \mathrm{div}\left(\rho_\varepsilon v_\varepsilon (c_\varepsilon)_i - \sum_{j=1}^L D_{ij}(c_\varepsilon) \nabla (c_\varepsilon)_j - m_i(c_\varepsilon) \nabla \phi_\varepsilon \right) = r_i(c_\varepsilon) \, .
\end{aligned}$$

We also regularize the initial condition. More precisely we introduce sequence $\rho_{0,\varepsilon} \in W^{1,2}(\Omega)$ such that $\rho_{0,\varepsilon} \to \rho_0$ in L^∞as $\varepsilon \to 0$. In the limit passage from the discretized system we will not be able to identify neither the term $\varepsilon \nabla \rho_\varepsilon \nabla v_\varepsilon$ nor the term $\varepsilon (c_\varepsilon)_i \Delta \rho$. However, the only information we need to know about the terms in limit passage with $\varepsilon \to 0$ are the apriori estimates. This motivates the following definition.

Definition 3.3. As a solution to the approximated problem we will call

- v_ε, c_ε and ϕ_ε as in definition 3.1,
- $\rho \in L^\infty(Q) \cap L^2(I; W^{1,2}(\Omega))$,
- $f_\varepsilon^1, f_\varepsilon^2 \in L^1$, $\sqrt{\varepsilon}\,\|f_\varepsilon^1\|_{L^1(\Omega)} + \sqrt{\varepsilon}\,\|f_\varepsilon^2\|_{L^1(\Omega)} \le C$

satisfying (3.4) and

$$-\int_Q \rho_\varepsilon \frac{\partial \varphi}{\partial t} \mathrm{d}x\mathrm{d}t - \int_Q \rho_\varepsilon v_\varepsilon \nabla\varphi \mathrm{d}x\mathrm{d}t + \varepsilon \int_Q \nabla\rho_\varepsilon \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_\Omega \rho_0 v_0 \varphi(0)\mathrm{d}x\,,$$

$$\begin{aligned}
&\int_Q \rho_\varepsilon v_\varepsilon \partial_t \varphi + (\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon + \mathbb{T}(c_\varepsilon \mathrm{D} v_\varepsilon) : \nabla\varphi \mathrm{d}x\mathrm{d}t + \varepsilon \int_Q f_\varepsilon^1 \varphi \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \rho_0 v_0 \varphi(0)\mathrm{d}x - \int_Q \rho_\varepsilon q(c_\varepsilon)\nabla\phi_\varepsilon \varphi \mathrm{d}x\mathrm{d}t\,,
\end{aligned}$$

$$\begin{aligned}
&\int_Q \rho(c_\varepsilon)_i \partial_t \varphi \mathrm{d}x\mathrm{d}t \\
&+ \int_Q (\rho(c_\varepsilon)_i v_\varepsilon + \sum_{j=1}^{L} D_{ij}(c_\varepsilon)\nabla(c_\varepsilon)_j + m_i(c_\varepsilon)\nabla\phi_\varepsilon + \varepsilon(c_\varepsilon)_i \nabla\rho) \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t \\
&+ \int_\Omega \varepsilon f_\varepsilon^2 \varphi \mathrm{d}x - \int_\Omega \rho(0)c(0)\varphi(0)\mathrm{d}x = \int_Q r_i(c_\varepsilon)\varphi \mathrm{d}x\mathrm{d}t
\end{aligned}$$

with every $\varphi \in C_0^\infty\left([0,T) \times \overline{\Omega}\right)$.

Next we discretize in time by implicit method. Because we need some version of maximum principles, we should not discretize Nernst-Planck and continuity equations in space. The other two equations we discretize also in space to make them better treatable.

Definition 3.4. As a solution to discretized problem we will call

- $\rho_{\varepsilon,n}^k \in W^{1,2}(\Omega)$,
- $v_{\varepsilon,n}^k \in V_{0,\mathrm{div}}^n$,
- $c_{\varepsilon,n}^k \in (W^{1,2}(\Omega))^L$, $\sum_{i=1}^L (c_{\varepsilon,n}^{k+1})_i = 1$,
- $\phi_{\varepsilon,n}^k \in V^n$

defined by relations:

$$\begin{aligned} \rho^0_{\varepsilon,n} &= \rho_{0,\varepsilon}\,, \\ v^0_{\varepsilon,n} &= P_{V^n_{0,\mathrm{div}}} v_0\,, \\ c^0_{\varepsilon,n} &= c_0\,, \end{aligned} \tag{3.5a}$$

$$\int_\Omega \epsilon \nabla \phi^0_{\varepsilon,n} \cdot \nabla \varphi_4 \mathrm{d}x - \int_{\partial\Omega} (\phi^0_{\varepsilon,n} - \phi_\Sigma)\varphi_4 \mathrm{d}x = \int_\Omega \rho^0_{\varepsilon,n} q(K(c^0_{\varepsilon,n}))\varphi_4 \mathrm{d}x\,, \tag{3.5b}$$

$$\begin{aligned} &\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n} v^{k+1}_{\varepsilon,n} - \rho^k_{\varepsilon,n} v^k_{\varepsilon,n}}{\tau^n} \cdot \varphi_1 \mathrm{d}x - \int_\Omega \rho^{k+1}_{\varepsilon,n} v^{k+1}_{\varepsilon,n} \otimes v^{k+1}_{\varepsilon,n} \cdot \nabla \varphi_1 \mathrm{d}x \\ &+ \varepsilon \int_\Omega \nabla \rho^{k+1}_{\varepsilon,n} \cdot \nabla v^{k+1}_{\varepsilon,n} \varphi_1 \mathrm{d}x \int_\Omega \mathbb{T}(K(c^{n+1}_{\varepsilon,n}), \mathrm{D}v^{n+1}_{\varepsilon,n}) \cdot \nabla \varphi \mathrm{d}x \\ &= \int_\Omega \rho^{k+1}_{\varepsilon,n} q(K(c^{n+1})_{\varepsilon,n}) \nabla \phi^{n+1}_{\varepsilon,n} \cdot \varphi_1 \mathrm{d}x\,, \end{aligned} \tag{3.5c}$$

$$\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n} - \rho^k_{\varepsilon,n}}{\tau^n} \varphi_2 \mathrm{d}x - \int_\Omega \rho^{k+1}_{\varepsilon,n} v^{k+1}_{\varepsilon,n} \cdot \nabla \varphi_2 \mathrm{d}x + \varepsilon \int_\Omega \nabla \rho^{k+1}_{\varepsilon,n} \cdot \nabla \varphi_2 \mathrm{d}x = 0\,, \tag{3.5d}$$

$$\begin{aligned} &\int_\Omega \rho^k_{\varepsilon,n} \frac{(c^{k+1}_{\varepsilon,n})_i - (c^k_{\varepsilon,n})_i}{\tau^n} \varphi_3 \mathrm{d}x + \int_\Omega \rho^{k+1}_{\varepsilon,n} v^{k+1}_{\varepsilon,n} \cdot \nabla (c^{k+1}_{\varepsilon,n})_i \varphi_3 \mathrm{d}x \\ &+ \int_\Omega (D_{ij}(K(c^{k+1}_{\varepsilon,n})) \nabla (c^{k+1}_{\varepsilon,n})_j + m_i(K(c^{k+1}_{\varepsilon,n})) \nabla \phi^{k+1}_{\varepsilon,n} \cdot \nabla \varphi_3 \mathrm{d}x \\ &= \int_\Omega r_i(K(c^{k+1}_{\varepsilon,n})) \varphi_3 \mathrm{d}\,, \end{aligned} \tag{3.5e}$$

$$\int_\Omega \epsilon \nabla \phi^{k+1}_{\varepsilon,n} \cdot \nabla \varphi_4 \mathrm{d}x - \int_{\partial\Omega} \epsilon(\phi^{k+1}_{\varepsilon,n} - \phi_\Sigma)\varphi_4 \mathrm{d}x = \int_\Omega \rho^{k+1}_{\varepsilon,n} q(K(c^{k+1}_{\varepsilon,n}))\varphi_4 \mathrm{d}x \tag{3.5f}$$

for $k = 0, \dots, N(\tau^n) - 1$ and every $\varphi_1 \in V^n_{0,\mathrm{div}}$, $\varphi_2 \in W^{1,2}(\Omega)$, $\varphi_3 \in W^{1,2}(\Omega)$ and$\varphi_4 \in V^n$.

The existence of an approximated solution will be proved by Shauder's fixed-point theorem. In the practice of numerical computation, the coupling will be done at each timestep separately before computing the next timestep.

For purposes of the analysis we need some bounds uniform in time. For simplicity of obtaining such bounds we first solve the decoupled equations in all timesteps and then we couple them simultaneously in all timesteps. This presents no further difficulty because for fixed τ^n there is only a finite number of timesteps.

Definition 3.5. As a solution to the decoupled problem with "old" quantities $\tilde{v}$ and $\tilde{c}$ we will call

- $\rho^k_{\varepsilon,n,\tilde{v},\tilde{c}} \in W^{1,2}(\Omega)$,
- $v^k_{\varepsilon,n,\tilde{v},\tilde{c}} \in V^n_{0,\mathrm{div}}$,
- $c^k_{\varepsilon,n,\tilde{v},\tilde{c}} \in (W^{1,2}(\Omega))^L$, $\sum_{i=1}^L c_i = 1$
- $\phi^k_{\varepsilon,n,\tilde{v},\tilde{c}} \in V^n$

indexed by $k = 0, \dots, N(\tau^n)$ and satisfying

$$\begin{aligned} \rho^0_{\varepsilon,n,\tilde{v},\tilde{c}} &= \rho_{O,\varepsilon}\,, \\ v^0_{\varepsilon,n,\tilde{v},\tilde{c}} &= P_{V^n_{0,\mathrm{div}}} v_0\,, \\ c^0_{\varepsilon,n,\tilde{v},\tilde{c}} &= c_0\,, \end{aligned} \tag{3.6a}$$

$$\int_\Omega \epsilon \nabla \phi^0_{\varepsilon,n,\tilde{v},\tilde{c}} \cdot \nabla \varphi_4 \mathrm{d}x - \int_{\partial\Omega} (\phi^0_{\varepsilon,n,\tilde{v},\tilde{c}} - \phi_\Sigma)\varphi_4 \mathrm{d}x = \int_\Omega \rho^0_{\varepsilon,n,\tilde{v},\tilde{c}} q(K(\tilde{c}^0))\varphi_4 \mathrm{d}x\,, \tag{3.6b}$$

$$\begin{aligned} &\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho^k_{\varepsilon,n,\tilde{v},\tilde{c}} v^k_{\varepsilon,n,\tilde{v},\tilde{c}}}{\tau^n} \cdot \varphi_1 \mathrm{d}x - \int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \tilde{v}^{k+1} \otimes v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \cdot \nabla \varphi_1 \mathrm{d}x \\ &+ \int_\Omega \nabla \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \cdot \nabla v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \varphi_1 \mathrm{d}x + \int_\Omega \mathbb{T}(K(\tilde{c}^{k+1}), \mathrm{D}v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}) \cdot \nabla \varphi \mathrm{d}x \\ &= \int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} q(K(\tilde{c}^{k+1})) \nabla \phi^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \cdot \varphi_1 \mathrm{d}x \end{aligned} \tag{3.6c}$$

$$\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho^k_{\varepsilon,n,\tilde{c},\tilde{v}}}{\tau^n} \varphi_2 \mathrm{d}x - \int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \tilde{v}^{k+1} \cdot \nabla \varphi_2 \mathrm{d}x + \int_\Omega \nabla \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \cdot \nabla \varphi_2 \mathrm{d}x = 0 \tag{3.6d}$$

$$\int_\Omega \rho^k_{\varepsilon,n,\tilde v,\tilde c}\frac{(c^{k+1}_{\varepsilon,n,\tilde v,\tilde c})_i-(c^k_{\varepsilon,n,\tilde v,\tilde c})_i}{\tau^n}\varphi_3\mathrm{d}x+\int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde v,\tilde c}v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\cdot\nabla(c^{k+1}_{\varepsilon,n,\tilde v,\tilde c})_i\varphi_3\mathrm{d}x$$
$$+\int_\Omega (D_{ij}(K(\tilde c^{k+1}))\nabla(c^{k+1}_{\varepsilon,n,\tilde v,\tilde c})_j+m_i(K(\tilde c^{k+1}))\nabla\phi^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\cdot\nabla\varphi_3\mathrm{d}x$$
$$=\int_\Omega r_i(K(\tilde c^{k+1}))\varphi_3\mathrm{d}x \tag{3.6e}$$

$$\int_\Omega \epsilon\nabla\phi^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\cdot\nabla\varphi_4\mathrm{d}x-\int_{\partial\Omega}(\phi^{k+1}_{\varepsilon,n,\tilde v,\tilde c}-\phi_\Sigma)\varphi_4\mathrm{d}x=\int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde v,\tilde c}q(K(\tilde c^{k+1}))\varphi_4\mathrm{d}x \tag{3.6f}$$

for $k=0,\dots,N(\tau^n)$ and every $\varphi_1\in V^n_{0,\mathrm{div}}$, $\varphi_2\in W^{1,2}(\Omega)$, $\varphi_3\in W^{1,2}(\Omega)$ and $\varphi_4\in V^n$.

Lemma 3.6. *Let* $\sup_k\left\|\tilde v^k\right\|_{V^n}\le K$, $\sum_{i=1}^L(c_0)_i=1$ *and* $\rho_m\le\rho_{0,\varepsilon}\le\rho_M$. *Then there exists* $\tilde\tau(n,\varepsilon)$ *depending on* K *and the choice of spaces* $V^n_{0,\mathrm{div}}$ *and* V^n*. such that if* $\tau_n\le\tilde\tau(n)$ *then there exists a solution to decoupled problem as defined in definition 3.5.*

Proof. We could prove the existence of solution by simple using the Lax-Milgram lemma. However, by this way we would not obtain uniform bound on $\|\nabla\rho\|_{L^2(\Omega)}$ needed for estimate of $\tilde\tau(n)$. That why we prove the existence using the Galerkin approximation. For the Galerkin approximation we need finite dimensional spaces generated by eigenfunctions of the Laplace operator to be able to test by $\Delta\rho$. The existence of Galerkin approximations is done by Lax-Milgramm lemma with bi-linear form

$$a_1(\rho,\varphi_2)=\int_\Omega\frac{\rho}{\tau}\varphi_2\mathrm{d}x-\int_\Omega\rho\tilde v\cdot\nabla\varphi_2\mathrm{d}x+\varepsilon\int_\Omega\nabla\rho\cdot\nabla\varphi_1\mathrm{d}x$$

which satisfies

$$a_1(\rho,\rho)\ge\frac{1}{\tau}\|\rho\|^2_{L^2(\Omega)}+\varepsilon\|\nabla\rho\|^2_{L^2(\Omega)}$$

because of the computation

$$\int_\Omega\rho v\cdot\nabla\rho\mathrm{d}x=\frac{1}{2}\int_\Omega v\cdot\nabla\rho^2\mathrm{d}x=0\,.$$

The crucial estimate we need is obtained by testing the continuity equation by $\Delta\rho^{k+1}$ (we for sake of simplicity omit the other indices). The convective term will be handled in the following way

$$\int_\Omega \operatorname{div}(\rho^{k+1}\tilde{v}^{k+1})\Delta\rho \mathrm{d}x \leq K \|\nabla\rho\|_{L^2(\Omega)} \left\|\Delta\rho^{k+1}\right\|_{L^2(\Omega)}$$
$$\leq \frac{\varepsilon}{2} \left\|\Delta\rho^{k+1}\right\|_{L^2(\Omega)} + C_\varepsilon K^2 \left\|\nabla\rho^{k+1}\right\|^2_{L^2(\Omega)} .$$

We may conclude that

$$\frac{1}{2\tau^n} \left\|\nabla\rho^{k+1}\right\|_{L^2(\Omega)} + \frac{1}{2\tau^n} \left\|\nabla(\rho^{k+1} - \rho^k)\right\|_{L^2(\Omega)} + \frac{\varepsilon}{2} \|\Delta\rho\|_{L^2(\Omega)}$$
$$\leq C_\varepsilon K^2 \left\|\nabla\rho^{k+1}\right\|_{L^2(\Omega)} + \frac{1}{2\tau^n} \left\|\nabla\rho^k\right\|_{L^2(\Omega)}$$

Summing equations and using the discrete Gronwall inequality gives us that there exists K_1 independent of choice of $\tau < \frac{1}{2C_\varepsilon K^2}$ such that $\left\|\nabla\rho^k\right\|_{L^2(\Omega)} \leq K_1 \|\nabla\rho_{0,\varepsilon}\|^2_{L^2(\Omega)}$. For the Poisson equation we use the Lax-Milgram lemma in a standard way. For the Navier-Stokes equations we shall use again the Lax-Milgram lemma together with estimates

$$\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} v}{\tau^n} \cdot v \mathrm{d}x \geq \frac{\rho_m}{\tau^n} \|v\|_{L^2(\Omega)} ,$$

$$-\int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \tilde{v}^{k+1} \otimes v \cdot \nabla v \mathrm{d}x \geq -\rho_M K \|v\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$
$$\geq -C_\beta \rho_M^2 K^2 \|v\|^2_{L^2(\Omega)} - \frac{\beta}{2} \|\nabla v\|^2_{L^2(\Omega)}$$

$$\int_\Omega \mathbb{T}(K(\tilde{c}^{k+1}), \mathrm{D}v) \cdot \nabla v \mathrm{d}x \geq \beta \|\nabla v\|_{L^2(\Omega)}$$

$$\int_\Omega \nabla\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \cdot \nabla v v \mathrm{d}x \geq -\left\|\nabla\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \|v\|_{V^n}$$
$$\geq -C_\beta \left\|\nabla\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right\|^2_{L^2(\Omega)} \|v\|^2_{L^2(\Omega)} - \frac{\beta}{4} \|\nabla v\|_{L^2(\Omega)}$$

where we have used the Korn's inequality, the minimum principle for the continuity equation (lemma 3.7) and equivalence of norms on finite dimensional subspace. For satisfying assumptions of the Lax-Milgram lemma we must choose

$$\tau < \frac{\rho_m}{\rho_M^2 K^2 C_\beta + C_\beta K_1 \|\nabla\rho_{0,\varepsilon}\|^2_{L^2(\Omega)}} \leq \frac{\rho_m}{\rho_M^2 K^2 c_\beta + C_\beta \left\|\nabla\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right\|^2_{L^2(\Omega)}} .$$

The treatment of Nernst-Planck equation is more complicated. We first shall prove that we are allowed to search the solution in form $\sigma := \sum_{i=1}^{L} c_i = 1$. By summing all equations for concentrations we get an equation for σ

$$\int_\Omega \rho^k_{\varepsilon,n} \frac{\sigma - 1}{\tau^n} \varphi_2 \mathrm{d}x + \int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \nabla\sigma \cdot \tilde{v} \varphi_2 \mathrm{d}x \,.$$

We see that the equations admit the solution $\sigma = 1$. Now we use the Lax-Milgramm lemma again. By the relation $c_L = 1 - \sum_{i=1}^{L-1} c_i$ we introduce a bi-linear form $a_2 : (W^m)^{L-1} \times (W^m)^{L-1} \to \mathbb{R}$ by formula:

$$a_2(c,\varphi) = \sum_{i=1}^{L} \Big(\int_\Omega \rho^k_{\varepsilon,n} \frac{c_i}{\tau^n} \varphi_i \mathrm{d}x + \int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{c},\tilde{v}} v^{k+1}_{\varepsilon,n,\tilde{c},\tilde{v}} \cdot \nabla c_i \varphi_i \mathrm{d}x$$
$$+ \int_\Omega (D_{ij}(K(\tilde{c}^{k+1}))\nabla c_j \mathrm{d}x \Big)$$

For the coercivity let us note the following estimates:

$$\sum_{i=1}^{L} \int_\Omega \rho^k_{\varepsilon,n\tilde{v},\tilde{c}} \frac{c_i}{\tau^n} c_i \mathrm{d}x \geq \frac{\rho_m}{\tau^n} \|c\|^2_{L^2(\Omega)} \,,$$

$$\sum_{i=1}^{L} \int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \mathrm{d}x \geq - \left\| v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \right\|_{V^n} \|c\|_{L^2(\Omega)} \|\nabla c\|_{L^2(\Omega)}$$
$$\geq -\frac{\alpha}{4} \|\nabla c\|^2_{L^2(\Omega)} - C_\alpha K_2^2 \|c\|^2_{L^2(\Omega)} \,,$$

$$\sum_{i=1}^{L} \int_\Omega D_{ij}(K(\tilde{c}^{k+1}))\nabla c_j \nabla c_i \mathrm{d}x \geq \alpha \|\nabla c\|_2^2$$

where for estimating the velocity in the convective term we used the lemma 3.8. For using Lax-Milgram lemma we need $\tau < \frac{\rho_m}{C_\alpha K_2^2}$. □

Lemma 3.7. *Let $\rho_m \leq \rho^k_{\varepsilon,n} \leq \rho_M$ and let $\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \in W^{1,2}(\Omega)$ such that (3.6d) holds. Moreover let $\operatorname{div} \tilde{v} = 0$. Then $\rho_m \leq \rho_{k+1} \leq \rho_M$.*

Proof. We rewrite the equation in form

$$\int_\Omega \frac{(\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m) - (\rho^k_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)}{\tau^n} \varphi_2 \mathrm{d}x - \int_\Omega \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \tilde{v}^{k+1} \cdot \nabla\varphi_2 \mathrm{d}x$$
$$+\varepsilon \int_\Omega \nabla(\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m) \cdot \nabla\varphi_2 \mathrm{d}x = 0$$

and use $(\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)^-$ as a test-function. We get

$$\frac{1}{\tau^n}\left\|(\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)^-\right\|_2^2 + \left\|\nabla(\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)^-\right\|_2^2 \leq 0\,.$$

While getting the result we shall use that $\rho^k_{\varepsilon,n,\tilde{v},\tilde{c}} \geq \rho_m$ and $x^- = \min(x,0) \leq 0$ to get $\int_\Omega -\frac{1}{\tau^n}(\rho^k_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)(\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)^-\mathrm{d}x \geq 0$ and

$$\int_\Omega \tilde{v}^{k+1}\nabla((\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)^-)^2\mathrm{d}x = \int_\Omega \operatorname{div}\tilde{v}^{k+1}((\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho_m)^-)^2\mathrm{d}x = 0\,.$$

□

Lemma 3.8. *For $v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}$ and $c^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}$ from definition 3.5 we have*

$$\left\|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right\|_{V^n} + \left\|c^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right\|_{W^{1,2}(\Omega)} \leq K_2$$

where K_2 is independent of choice of $\tilde{v}$ and $\tilde{c}$.

Proof. From the lemma 3.7 we have $\left\|\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right\|_{L^\infty(\Omega)} \leq \rho_M$ and from the definition of the retract we conclude $\|K(\tilde{c})\|_{L^\infty(\Omega)} \leq 1$. As a consequence we also have $\nabla\phi^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}$ uniformly bounded. Next we shall test the continuity equation by $\frac{1}{2}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2$ and compute

$$\begin{aligned}
&\frac{\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 - \rho^k_{\varepsilon,n}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2}{2\tau} = \\
&\frac{2\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 - 2\rho^k_{\varepsilon,n}v^k_{\varepsilon,n}\cdot v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - \rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 + \rho^k_{\varepsilon,n}\left|v^k_{\varepsilon,n}\right|^2}{2\tau^n} \\
&- \frac{\rho^k_{\varepsilon,n}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 - 2\rho^k_{\varepsilon,n}v^k_{\varepsilon,n}\cdot v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} + \rho^k_{\varepsilon,n}\left|v^k_{\varepsilon,n}\right|^2}{2\tau^n} = \\
&\frac{2\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 - 2\rho^k_{\varepsilon,n}v^k_{\varepsilon,n}\cdot v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}}{2\tau^n} \\
&- \frac{\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 - \rho^k_{\varepsilon,n}\left|v^k_{\varepsilon,n}\right|^2 + \rho_k\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - v^k_{\varepsilon,n}\right|^2}{2\tau^n}
\end{aligned}$$

So we have the following equality

$$\begin{aligned}
\int_\Omega &\frac{\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 - \rho^k_{\varepsilon,n}v^k_{\varepsilon,n}\cdot v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}}{\tau^n} + (\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\otimes\tilde{v})\cdot\nabla v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} \\
&\qquad + \varepsilon\nabla\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\cdot\nabla v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\mathrm{d}x \qquad (3.7) \\
&= \int_\Omega \frac{\rho^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}}\right|^2 - \rho^k_{\varepsilon,n}\left|v^k_{\varepsilon,n}\right|^2 + \rho^k_{\varepsilon,n}\left|v^{k+1}_{\varepsilon,n,\tilde{v},\tilde{c}} - v^k_{\varepsilon,n}\right|^2}{2\tau^n}
\end{aligned}$$

Now we just test the Navier-Stokes equation by $v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}$ and get

$$\begin{aligned}
&\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\left|v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\right|^2-\rho^{k}_{\varepsilon,n}\left|v^{k}_{\varepsilon,n}\right|^2+\rho^{k}_{\varepsilon,n}\left|v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}-v^{k}_{\varepsilon,n}\right|^2}{2\tau^n}\\
&+\mathbb{T}(\tilde c,\mathrm{D}v^{k+1})_{\varepsilon,n,\tilde v,\tilde c}:\nabla v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\mathrm{d}x=\int_\Omega \rho q(K(\tilde c))\nabla\phi^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\cdot v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}\mathrm{d}x
\end{aligned} \tag{3.8}$$

Using Hölder and discrete Gronwall inequalities concludes the statement about $v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}$. For the statement about $c^{k+1}_{\varepsilon,n,\tilde v,\tilde c}$ we use the same estimates as while proving lemma 3.6. □

Lemma 3.9. *There exists $\tilde\tau(n)$ depending on choice of spaces V^n, W^n such that if $\tau_n \le \tilde\tau(n)$ there exists a solution to the discrete problem as defined in definition 3.4.*

Proof. We use Shauder fixed-point theorem. Using lemma 3.6 with K from lemma 3.8 we define mapping $\mathcal{F} : B_K \subset V^m \times L^2(\Omega)^2 \to B_K$ which to $(\tilde v, \tilde c, \tilde\rho)$ assigns $v^{k+1}_{\varepsilon,n,\tilde v,\tilde c}$ and $c^{k+1}_{\varepsilon,n,\tilde v,\tilde c}$ solving the decoupled problem. As we know from lemma 3.8, we also have $\|\mathcal{F}(B_K)\|_{V^n\times W^{1,2}(\Omega)} \le K$ and hence $\mathcal{F}(B)$ is compact. The last thing we need to assure is continuity of $\mathcal{F}$. Let $(\tilde v^m, \tilde c^m) \to (\tilde v, \tilde c)$. We will for simplicity define $(v^m, c^m) := \mathcal{F}(\tilde v^m, \tilde c^m)$. From the same reason we abbreviate solutions of decoupled continuity equation as ρ^m and solutions of decoupled Poisson equation as ϕ^m. From apriori bounds already obtained we may choose subsequences such that

- $v^m \to v$ in $V^n_{0,\mathrm{div}}$,
- $c^m \rightharpoonup c$ weakly in $W^{1,2}(\Omega)$,
- $c^m \to c$ strongly in $L^2(\Omega)$,
- $\phi^m \to \phi$ strongly in V^n,
- $\rho^m \rightharpoonup^* \rho$ weakly-* in $L^\infty(\Omega)$,
- $\rho^m \rightharpoonup \rho$ in $W^{1,2}(\Omega)$.

The obtained convergences are enough to converge in equations and obtain continuity of of the mapping $\mathcal{F}$. □

Lemma 3.10. *Let $(c^k_{\varepsilon,n})_i \ge 0$ almost everywhere and $c^{k+1}_{\varepsilon,n}$ from the definition 3.4. Then $(c^{k+1}_{\varepsilon,n})_i \ge 0$ almost everywhere.*

Proof. The proof of this goes in lines of proof of maximum principle for elliptic partial differential equations done in the book [GT01]. As the theorem 8.1 of [GT01] is not directly applicable to our problem the proof will be rewritten here. For the contradiction let us assume that $m = \text{essinf}_\Omega (c_{\varepsilon,n}^{k+1})_i < 0$. Let us take arbitrary $r \in (m, 0)$ and test the equation by $((c_{\varepsilon,n}^{k+1})_i - r)^-$. Some terms in the equation have the right sign sign and we may forget them:

$$\int_\Omega \frac{-\rho_{\varepsilon,n}^k (c_{\varepsilon,n}^k)_i}{\tau^n} ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x \geq 0\,,$$

$$\int_\Omega m_i (c_{\varepsilon,n}^{k+1}) \nabla \phi_{\varepsilon,n}^{k+1} \cdot \nabla ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x = 0\,,$$

$$\int_\Omega r_i (c_{\varepsilon,n}^{k+1}) ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x \leq 0\,.$$

We may estimate the norm $\left\| (c_{\varepsilon,n}^{k+1} - k)_i^- \right\|_{W^{1,2}(\Omega)}$: by the time derivative and the diffusive term:

$$\begin{aligned}
&\int_\Omega \rho_{\varepsilon,n}^k \frac{(c_{\varepsilon,n}^{k+1})_i}{\tau} ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x \\
&= \int_\Omega \rho_{\varepsilon,n}^k \frac{((c_{\varepsilon,n}^{k+1})_i - r)^-}{\tau^n} ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x + \int_\Omega \rho_{\varepsilon,n}^k \frac{r}{\tau^n} ((c_{\varepsilon,n}^{k+1})_i - r) \mathrm{d}x \\
&\geq C \left\| ((c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{L^2(\Omega)}\,,
\end{aligned}$$

$$\int_\Omega \sum_{j=1}^{L} (D_{ij} (c_{\varepsilon,n}^{k+1}) \nabla (c_{\varepsilon,n}^{k+1})_j) \nabla ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x \geq C \left\| \nabla ((c_{\varepsilon,n}^{k+1})_i - r)^-) \right\|_{L^2(\Omega)}\,.$$

Finally the only term for estimating is the convective term:

$$\begin{aligned}
\left\| ((c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{W^{1,2}(\Omega)}^2 &\leq C \int_\Omega \rho_{\varepsilon,n}^{k+1} v_{\varepsilon,n}^{k+1} \cdot \nabla (c_{\varepsilon,n}^{k+1})_i ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x \\
&\leq \int_{\nabla ((c_{\varepsilon,n}^{k+1})_i - r)^- \neq 0} \rho_{\varepsilon,n}^{k+1} v_{\varepsilon,n}^{k+1} \cdot \nabla ((c_{\varepsilon,n}^{k+1})_i - r)^- ((c_{\varepsilon,n}^{k+1})_i - r)^- \mathrm{d}x \\
&\leq C_1 \left\| (c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{L^2\left(\{\nabla ((c_{\varepsilon,n}^{k+1})_i - r)^- \neq 0\}\right)} \left\| (c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{W^{1,2}(\Omega)}\,.
\end{aligned}$$

Because $\left\| (c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{W^{1,2}(\Omega)} \neq 0$ we may divide by it and get

$$\begin{aligned}
\left\| ((c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{W^{1,2}(\Omega)} &\leq C \left\| (c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{L^2\left(\{\nabla ((c_{\varepsilon,n}^{k+1})_i - r)^- \neq 0\}\right)} \\
&\leq \left| \{\nabla ((c_{\varepsilon,n}^{k+1})_i - r)^- \neq 0\} \right| \left\| ((c_{\varepsilon,n}^{k+1})_i - r)^- \right\|_{L^2(\Omega)}
\end{aligned}$$

and conclude that $\left|\{\nabla((c_{\varepsilon,n}^{k+1})_i - r)^- \neq 0\}\right| \geq C$ independently of choice r. This tells us that $(c_{\varepsilon,n}^{k+1})_i$ attains its maximum on set of nonzero measure. By lemma 7.7 of [GT01] we have on this set $\nabla(c_{\varepsilon,n}^{k+1})_i = 0$. Now we arrive to contradiction with fact that $\left|\{\nabla((c_{\varepsilon,n}^{k+1})_i - r)^- \neq 0\}\right| \geq C$ if $r \to m$. $\square$

Now we construct piecewise constant approximations of the solution. Now let us define $f_{n,\varepsilon}(t) = f_{\varepsilon,n}^k$ for $t \in ((k-1)\tau; k\tau]$, where f is selected from ρ, v, c, ϕ. Moreover we need some notation: $\Delta_\tau^+ f(t) = f(t+\tau) - f(t)$, $\Delta_\tau^- f(t) = f(t) - f(t-\tau)$, $\mathcal{T}_\tau f(t) = f(t-\tau)$, $\partial_{t,\tau}^{\pm} f(t) = \frac{\Delta_\tau^{\pm} f(t)}{\tau}$. We define also $\tilde{v}_{\varepsilon,n}$, $\tilde{c}_{\varepsilon,n}$, $\widetilde{\rho v}_{\varepsilon,n}$ and $\widetilde{\rho c}_{\varepsilon,n}$ as linear approximations of corresponding quantities.

Lemma 3.11. *We have the following bounds:*

- $\|\rho_{\varepsilon,n}\|_{L^\infty(Q)} + \varepsilon^{\frac{1}{2}} \|\nabla \rho_{\varepsilon,n}\|_{L^2(Q)} \leq C,$
- $\|v_{\varepsilon,n}\|_{L^\infty(I;L^2(\Omega))} + \|\nabla v_{\varepsilon,n}\|_{L^2(Q)} \leq C,$
- $\|c_{\varepsilon,n}\|_{L^\infty(Q)} + \|\nabla c_{\varepsilon,n}\|_{L^2(Q)} \leq C,$
- $\|\phi_{\varepsilon,n}\|_{L^\infty(I;W^{1,2}(\Omega))} \leq C,$
- $\|\tilde{c}_{\varepsilon,n}\|_{L^\infty(Q)} + \|\tilde{c}_{\varepsilon,n}\|_{L^2(I;W^{1,2}(\Omega))} + \sqrt{\tau} \left\|\frac{\partial \tilde{c}_{\varepsilon,n}}{\partial t}\right\|_{L^2(Q)} \leq C,$
- $\|\tilde{v_{\varepsilon,n}}\|_{L^\infty(\infty;L^2(Q))} + \|\tilde{v}_{\varepsilon,n}\|_{L^2(I;W^{1,2}(\Omega))} + \sqrt{\tau} \left\|\frac{\partial \tilde{v}_{\varepsilon,n}}{\partial t}\right\|_{L^2(Q)} \leq C.$

Proof. The estimate $\|\rho_{\varepsilon,n}\|_{L^\infty(Q)}$ is a direct consequence of lemma 3.7. The bound on $\|c_{\varepsilon,n}\|_{L^\infty(Q)}$ is obtained from lemma 3.10 and the fact that we constructed concentrations in such way that $\sum_{i=1}^L (c_{\varepsilon,n})_i = 1$. Now we are able to deduce the bound on $\phi_{\varepsilon,n}$. Bounds $\|v\|_{L^\infty(\infty;L^2(Q))}$, $\|v\|_{L^2(I;W^{1,2}(\Omega))}$ and $\sqrt{\tau}\left\|\frac{\partial v}{\partial t}\right\|_{L^2(Q)}$ are obtained from equality 3.8 using Hölder, Young and discrete Gronwall inequalities. Testing the continuity equation by $\rho_{\varepsilon,n}$, using

$$\int_\Omega \rho_{\varepsilon,n} v_{\varepsilon,n} \cdot \nabla \rho_{\varepsilon,n} \mathrm{d}x = \frac{1}{2} \int_\Omega v_{\varepsilon,n} \cdot \nabla \rho_{\varepsilon,n}^2 \mathrm{d}x = 0$$

and L^∞bounds on ρ we get $\varepsilon^{\frac{1}{2}} \|\nabla \rho_{\varepsilon,n}\|_{L^2(Q)} \leq C$. For the last estimates let us test the discretized Nernst-Planck equation by $c_{\varepsilon,m}^{k+1}$ to get:

$$\frac{\rho_m \left\|c_{\varepsilon,n}^{k+1}\right\|_{L^2(\Omega)} + \rho_m \left\|c_{\varepsilon,n}^{k+1} - c_{\varepsilon,n}^k\right\|_{L^2(\Omega)} - \rho_M \left\|c_{\varepsilon,n}^k\right\|_{L^2(\Omega)}}{2\tau^n} + \alpha \left\|\nabla c_{\varepsilon,n}^{k+1}\right\|_{L^2(\Omega)}$$
$$\leq \int_\Omega \rho_{\varepsilon,n}^{k+1} v_{\varepsilon,n}^{k+1} c_{\varepsilon,n}^{k+1} \cdot \nabla c_{\varepsilon,n}^{k+1} + m(c_{\varepsilon,n}^{k+1}) \nabla \phi_{\varepsilon,n}^{k+1} \cdot \nabla c_{\varepsilon,n}^{k+1} + r(c_{\varepsilon,n}^{k+1}) c_{\varepsilon,n}^{k+1} \mathrm{d}x\,.$$

Using Hölder and Young inequalities together with already known estimates gives us bounds on $\|\nabla c_{\varepsilon,n}\|_{L^2(Q)}$ and $\sqrt{\tau}\left\|\frac{\partial \tilde{c}_{\varepsilon,n}}{\partial t}\right\|_{L^2(Q)}$. □

Now let us derive relations satisfied by the approximations. First we work a bit with the Nernst-Planck equation. If we test the continuity equation with $(c^{k+1}_{\varepsilon,n})_i\varphi$ we get

$$\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n}(c^{k+1}_{\varepsilon,n})_i - \rho^k_{\varepsilon,n}(c^{k+1}_{\varepsilon})_i}{\tau^n}\varphi \mathrm{d}x$$
$$- \int_\Omega \rho^{k+1}_{\varepsilon,n} v^{k+1}_{\varepsilon,n}\cdot\nabla(c^{k+1}_{\varepsilon,n})_i\varphi\mathrm{d}x - \int_\Omega \rho^{k+1}_{\varepsilon,n}(c^{k+1}_{\varepsilon})_i v^{k+1}_{\varepsilon}\cdot\nabla\varphi\mathrm{d}x$$
$$+ \int_\Omega \varepsilon\nabla\rho^{k+1}_{\varepsilon,n}\cdot\nabla(c^{k+1}_{\varepsilon,n})_i\varphi\mathrm{d}x + \int_\Omega \varepsilon\nabla\rho^{k+1}_{\varepsilon,n}(c^{k+1}_{\varepsilon,n})_i\cdot\nabla\varphi\mathrm{d}x = 0\,.$$

Inserting this into Nernst-Planck equations we obtain

$$\int_\Omega \frac{\rho^{k+1}_{\varepsilon,n}(c^{k+1}_{\varepsilon,n})_i - \rho^k_{\varepsilon,n}(c^k_{\varepsilon,n})_i}{\tau^n}\varphi_3 \mathrm{d}x - \int_\Omega \rho^{k+1}_{\varepsilon,n} v^{k+1}_{\varepsilon,n}(c^{k+1}_{\varepsilon,n})_i\nabla\varphi_3\mathrm{d}x$$
$$+ \int_\Omega \varepsilon\nabla\rho^{k+1}_{\varepsilon,n}\cdot\nabla(c^{k+1}_{\varepsilon,n})_i\varphi\mathrm{d}x + \int_\Omega \varepsilon\nabla\rho^{k+1}_{\varepsilon,n}(c^{k+1}_{\varepsilon,n})_i\cdot\nabla\varphi\mathrm{d}x$$
$$+ \int_\Omega (D_{ij}(c^{k+1}_{\varepsilon,n})\nabla(c^{k+1}_{\varepsilon,n})_j + m_i(c^{k+1}_{\varepsilon,n})\nabla\phi^{k+1}_{\varepsilon,n}\cdot\nabla\varphi_3\mathrm{d}x$$
$$= \int_\Omega r_i(K(c_{k+1}))\varphi_3\mathrm{d}x\,.$$

To get the relations for interpolants we test the equations by $\int_{k\tau^n}^{(k+1)\tau^n}\varphi\mathrm{d}t$ with φ chosen such that $\varphi(T)=0$ and do summation by parts. denoting by $(\cdot,\cdot)$ the L^2scalar product we get:

$$-(\rho_{\varepsilon,n}v_{\varepsilon,n},\partial^+_{t,\tau}\varphi)_Q - (\rho_{0,\varepsilon}v_0,\varphi(0))_\Omega - (\rho_{\varepsilon,n}v_{\varepsilon,n}\otimes v_{\varepsilon,n} - \mathbb{T}(c_{\varepsilon,n},\mathrm{D}v_{\varepsilon,n}),\nabla\varphi)_Q$$
$$+\varepsilon(\nabla\rho_{\varepsilon,n}\cdot\nabla v_{\varepsilon,n},\varphi)_Q = (\rho_{\varepsilon,n}q(c_{\varepsilon,n})\nabla\phi_{\varepsilon,n},\varphi)_Q\,,$$

$$(\rho_{\varepsilon,n},\partial^+_{t,\tau}\varphi)_Q + (\rho_{0,\varepsilon},\varphi(0))_\Omega + (\rho_{\varepsilon,n}v_{\varepsilon,n},\nabla\varphi)_Q = \varepsilon(\nabla\rho_{\varepsilon,n},\nabla\varphi^n)_Q\,,$$

$$-(\rho_{\varepsilon,n}(c_{\varepsilon,n})_i,\partial^+_{t,\tau}\varphi)_Q - (\rho_{0,\varepsilon}(c_0)_i,\varphi(0))_\Omega$$
$$+(\rho_{\varepsilon,n}(c_{\varepsilon,n})_i v_{\varepsilon,n} + \sum_{j=1}^L D_{ij}(c_{\varepsilon,n})\nabla(c_{\varepsilon,n})_j + m_i(c_{\varepsilon,n})\nabla\phi_{\varepsilon,n},\nabla\varphi)_Q$$
$$+\varepsilon(\nabla\rho_{\varepsilon,n}\cdot\nabla c_{\varepsilon,n},\varphi)_Q + \varepsilon(c_{\varepsilon,n}\nabla\rho_{\varepsilon,n})_Q = -(r_i(c_{\varepsilon,n}),\varphi)_Q\,,$$
$$(\nabla\phi_{\varepsilon,n},\nabla\varphi)_Q + (\phi_{\varepsilon,n}-\phi_\sigma)_Q = (\rho_{\varepsilon,n}q(c_{\varepsilon,n}),\varphi)_Q\,.$$

Lemma 3.12. *For linear approximations we have the following bounds:*

- $\left\|\widetilde{\rho v}_{\varepsilon,n}\right\|_{L^2\left(W^{2,2^*}\right)} \leq C$
- $\left\|\widetilde{\rho c}_{\varepsilon,n}\right\|_{L^2\left(W^{2,2^*}\right)} \leq C$
- $\|\widetilde{\rho}_{\varepsilon,n}\|_{L^2\left(W^{1,2^*}\right)} \leq C$

Proof. The proof of the lemma consist just of using the Hölder inequalities in weak formulations of the equations. □

Lemma 3.13. *There exists at least one approximated weak solution to the problem* (2.9) *as defined in the definition 3.3. Moreover estimates from the lemma 3.11 still hold.*

Proof. In the previous we constructed the approximate solutions $\rho_{\varepsilon,n}$, $v_{\varepsilon,n}$, $c_{\varepsilon,n}$ and $\phi_{\varepsilon,n}$. By apriori estimates we may choose convergent subsequences:

- $\rho_{\varepsilon,n} \rightharpoonup^* \rho$ in $L^\infty(Q)$,
- $\rho_{\varepsilon,n} \rightharpoonup \rho$ in $L^2(I; W^{1,2}(\Omega))$,
- $\nabla\rho_{\varepsilon,n}\nabla v_{\varepsilon,n} \rightharpoonup f_1$ in $L^1(Q)$,
- $\nabla\rho_{\varepsilon,n}\nabla c_{\varepsilon,n} \rightharpoonup f_2$ in $L^1(Q)$,
- $v_{\varepsilon,n} \rightharpoonup^* v$ in $L^\infty(I; L^2(\Omega))$,
- $v_{\varepsilon,n} \rightharpoonup v$ in $L^2(I; W^{1,2}(\Omega))$,
- $c_{\varepsilon,n} \rightharpoonup^* c$ in $L^\infty(Q)$,
- $c_{\varepsilon,n} \rightharpoonup c$ in $L^2(I; W^{1,2}(\Omega))$,
- $\phi_{\varepsilon,n} \rightharpoonup^* \phi$ in $L^\infty(I; W^{1,2}(\Omega))$,
- $\tilde{\rho v}_{\varepsilon,n} \rightharpoonup \tilde{\rho v}$ in $L^2(Q)$,
- $\tilde{\rho c}_{\varepsilon,n} \rightharpoonup \tilde{\rho c}$ in $L^2(Q)$.

Moreover because

$$\|\mathcal{T}_\tau c_{\varepsilon,n} - c_{\varepsilon,n}\|_2 = \tau \left\| \frac{\partial \tilde{c}_{\varepsilon,n}}{\partial t} \right\|_2$$

and

$$\|\mathcal{T}_\tau v_{\varepsilon,n} - v_{\varepsilon,n}\|_2 = \tau \left\| \frac{\partial \tilde{v}_{\varepsilon,n}}{\partial t} \right\|_2$$

we have convergences of the shifted quantities

- $\mathcal{T}_\tau v_{\varepsilon,n} \rightharpoonup^* v$ in $L^\infty(I; L^2(\Omega))$,
- $\mathcal{T}_\tau v_{\varepsilon,n} \rightharpoonup v$ in $L^2(I; W^{1,2}(\Omega))$,
- $\mathcal{T}_\tau c_{\varepsilon,n} \rightharpoonup^* c$ in $L^\infty(Q)$,
- $\mathcal{T}_\tau c_{\varepsilon,n} \rightharpoonup c$ in $L^2(I; W^{1,2}(\Omega))$.

Using the Aubin-Lions lemma we get

- $\rho_{\varepsilon,n} \to \rho$ in $L^2(Q)$,
- $\widetilde{\rho v}_{\varepsilon,n} \to \widetilde{\rho v}$ in $L^2\left(W^{1,2*}\right)$,
- $\widetilde{\rho c}_{\varepsilon,n} \to \widetilde{\rho c}$ in $L^2\left(W^{1,2*}\right)$.

By uniqueness of the distributional limit (because $c_{\varepsilon,n}\varphi \in L^2(W^{1,2})$ for $\varphi \in C_0^\infty((0,T)\times\Omega)$) we get $\widetilde{\rho v} = \rho v$ and $\widetilde{\rho c} = \rho c$. For strong convergence of concentrations we compute

$$\begin{aligned}\|c_{\varepsilon,n} - c\|_2^2 &\le C\int_\Omega \rho_{\varepsilon,n}(c_{\varepsilon,n} - c)^2 \mathrm{d}x \\ &= \int_\Omega \rho_{\varepsilon,n} c_{\varepsilon,n}(c_{\varepsilon,n} - c)\mathrm{d}x - \int_\Omega \rho_{\varepsilon,n} c_{\varepsilon,n} c \mathrm{d}x + \int_\Omega \rho^n c^2 \mathrm{d}x \to 0\end{aligned}$$

To converge we also need the strong convergence in the gradient of ϕ. From equation for electric potential we have:

$$\|\nabla\phi_{\varepsilon,n} - \nabla\phi\|_2 = (\rho_{\varepsilon,n} q(c_{\varepsilon,n}), \phi_{\varepsilon,n} - \phi) + (\nabla\phi, \nabla\phi_{\varepsilon,n} - \nabla\phi) \to 0\,.$$

Having these convergences in hand we see that limit functions satisfy equations (2.9). Also because $c_{\varepsilon,n}$ converge strongly in $L^2(Q)$, we may select a subsequence converging pointwise and pass to limit in the constraint $c_{\varepsilon,n} \ge 0$. Finally the statements about norms are direct consequence of lowersemicontinuity of norm. □

Theorem 3.14. *There exists at least one weak solution to the problem* (2.9).

Proof. The proof of this theorem is based on the same convergences as the proof of lemma 3.13. The main difference is that now $\rho_\varepsilon \to \rho$ only in $L^2\left(I; (W^{1,2}(\Omega))^*\right)$. Moreover we have from apriori bounds that

- $\varepsilon\nabla\rho_\varepsilon \to 0$ strongly in $L^2(Q)$
- $\varepsilon\nabla f_\varepsilon^1 \to 0$ strongly in $L^1(Q)$
- $\varepsilon\nabla f_\varepsilon^2 \to 0$ strongly in $L^1(Q)$

The solution of continuity equation is also a renormalized solution because $v \in L^2(I; W^{1,2}(\Omega))$ has automatically the renormalisation property as was proved in [DL89]. □

4 conclusion

The goal of the work was to remove not too realistic assumption that all species have the same mass densities and also remove volume additivity hypothesis from the works [Rou06], [Rou05] and [Rou07]. This goal was archived in sections 2 and 3. The proof presented in the section 3 is not usable for numerical purposes. It uses several limit passages to deal at least partly with maximum principles. One still could read a few ansatzes for numerical computation from it. One could get the conditions on timesteps, the regularization term in continuity equation and corresponding term in Navier-Stokes equation and the fact that for the density and concentrations one should have much richer spaces than for other variables. Up to author's knowledge the result presented in the thesis is new although it assembles in right way the techniques from articles [AF07] and [Rou07].

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